A CERTAIN OPEN MANIFOLD WHOSE GROUP IS UNITY

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[Received 29 May 1935]

1. In a recent paper,† which will be referred to as T.M., I attempted unsuccessfully‡ to prove that every finite, unbounded manifold in which every circuit bounds a 2-cell is a 3-sphere. On the basis of T.M., Theorem 1, which is false, I had prepared an article containing the theorem on infinite manifolds announced in T.M., § 1. An obvious corollary to this theorem is that any infinite, unbounded manifold,‖ in which every finite circuit bounds a 2-cell and every finite 2-cycle bounds a finite region, is what we shall call a *formal* 3-*cell*. In § 3 below, an example is given which disproves this.

By a formal 3-cell we shall mean an infinite, unbounded manifold, a sub-division†† of which, say C, contains an infinite sequence of elements E_1, E_2,..., such that E_{n+1} contains every solid of C which meets E_n. Under these conditions it is obvious that any solid, and therefore any finite region, in C is contained in E_n for some value of n. Moreover, it is not difficult to show that a subdivision of C has a rectilinear model covering Euclidean 3-space; also that the symbol for such a rectilinear complex is a formal 3-cell. Thus an infinite manifold is a formal 3-cell, if and only if its rectilinear model in Hilbert space is in (1, 1) semilinear correspondence with Euclidean 3-space.

To avoid verbal complications we shall not always distinguish in our notation or terminology between a manifold and one of its sub-divisions. Thus a manifold will mean an abstraction determined by the totality of symbols which are combinatorially equivalent‡‡ to a given symbolic manifold. Any symbolic manifold may be called a

† *Quart. J. of Math.* (Oxford), 5 (1934), 308–20. ‡ Ibid. 6 (1935).

‖ As in T.M., the words 'three-dimensional' will often be omitted.

†† In referring to a subdivision of an infinite manifold it is implied that any finite sub-complex has a finite subdivision. The theorems about combinatorial subdivisions which are relevant to this paper are to be found in articles by M. H. A. Newman (*J. of London Math. Soc.* 2 (1927), 56–64), and J. H. C. Whitehead (*Proc. Cambridge Phil. Soc.* 31 (1935), 69–75).

‡‡ These definitions, and many of the subsequent arguments, are based upon an article by J. W. Alexander, *Annals of Math.* 31 (1930), 292–320.

covering of the corresponding manifold, and in dealing with a sub-complex K, of a manifold M, we restrict ourselves to coverings which contain K as a sub-complex. More precisely, the transformation $(A, a)^{-1}$ shall not be applied to a covering of M unless $a(A)B$ belongs to K, where B is any component, including 1, such that aB belongs to K,† and (U) stands for the boundary of a complex U.

2. By a non-singular deformation of an i-dimensional manifold N_i ($i = 1, 2$, or 3), in a three-dimensional manifold M, we shall mean the resultant of a finite sequence of transformations of the form‡

$$N_i \to N_i + (E_{i+1})$$

if $i = 1$ or 2, where E_{i+1} is an $(i+1)$-element which meets N_i in an i-element on (E_{i+1}); and of the form

$$N_i \to N_i + E_i$$

if $i = 3$, where E_i is a 3-element whose boundary meets (N_i) in a 2-element and which is either contained in N_i or has no internal component in common with N_i.

LEMMA 1. *If a circuit c, on a two-dimensional manifold S, in M, is transformable into c' by a non-singular deformation, there is a non-singular deformation of S which carries c into c'.*

Let
$$c \to c + (E_2)$$
be the first step in the deformation of c. Let (E_2) meet c in a segment l and let
$$m = (E_2) - l.$$

After a slight deformation, the intersection $S.E_2$, if it exists, will consist of non-singular circuits and segments, the latter having their end-points on m, and S will not touch E_2, except along the boundary segment l. At least one of these circuits, say c_1, can be joined to a vertex on m by a non-singular segment t, on E_2, which does not meet S except in the end-point on c_1. Let p be this end-point and let

$$N(K, L)$$

stand for the aggregate of components in any symbolic complex L which meet a sub-complex K. With a suitable covering, $N(t, M)$ is a 3-element and $N(p, S)$ a 2-element which divides $N(t, M)$ into two

† Unity stands for the 'empty' complex, and the convention is that $L1 = L$, L being any complex.

‡ Addition is to modulus 2 throughout this paper, though we shall sometimes use the minus sign.

3-elements. Let E_3 be the one which contains t. Then c_1 is transformed into a segment by the deformation

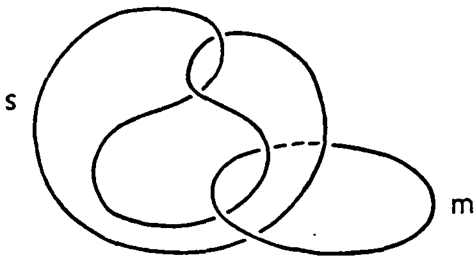
$$S \rightarrow S+(E_3).$$

In this way all the circults in $S.E_2$ can be eliminated.

When the circuits have been eliminated there will be at least one segment in $S.E_2$, say u, which, together with a segment of m, bounds a 2-element C_2, on E_2, containing no other component of S. If E_3 is defined as before, with $N(C_2, M)$ and $N(u, S)$ taking the place of $N(t, M)$ and $N(p, S)$, the segment u is eliminated from $S.E_2$ by the deformation

$$S \rightarrow S+(E_3).$$

Reiterating this process, we obtain a non-singular deformation of



S into a surface S^1, which does not meet E_2 except in l, and this deformation leaves c unaltered. It is now obvious that the first step in the deformation $c \rightarrow c'$ can be realized by a non-singular deformation of S^1, and the lemma follows from induction on the number of steps in $c \rightarrow c'$.

If a circuit in M is contained in a 3-element in M it will be called an *elementary circuit*. A circuit which bounds a (singular) 2-cell but which is not an elementary circuit will be called a *self-linking* circuit. The simplest type of self-linking circuit is illustrated by the diagram, the manifold being the residual space of a circuit m in Euclidean space, and s being a self-linking circuit.

We shall need two lemmas about punctured spheres.† We first recall from T.M. pp. 319–20, that any two punctured spheres are equivalent if they have the same number of boundary 2-spheres.‡

† Cf. T. M. § 2. When we refer to a punctured sphere or to any other bounded manifold, it is to be assumed that the boundary is non-singular.

‡ The argument used in T.M. is valid in virtue of Alexander's theorem about the separation of a 3-sphere by a 2-sphere (*Proc. National Ac. of Sci.* 10 (1924), 6–8).

LEMMA 2. *Any complex K, in a punctured sphere U, is contained in an element which is contained in U, provided K does not separate any pair of 2-spheres in (U).*

The 2-spheres in (U) can be connected by a system of tubes which do not meet K, in such a way as to form a single 2-sphere which bounds a 3-element contained in U and containing K.

LEMMA 3. *Let U be a punctured sphere and E a 3-element whose boundary meets (U) in a band bounded by a pair of non-singular and non-intersecting circuits, and either*

(i) *let E have no other component in common with U,*

or (ii) *let E be contained in U.*

Then $U+E$ is a punctured sphere (case i) *or a pair of punctured spheres* (case ii).

Taking U to be actually imbedded in a 3-sphere the proof of the second case is obvious. In the first case one can, by the standard method of starring elements, imbed E in the 3-element bounded by the 2-sphere in (U) which meets (E), and the proof is again obvious.

From Lemma 1 it follows that any circuit which is contained in a punctured sphere in a given manifold is an elementary circuit.

Now let σ be a non-singular circuit in a manifold M, and let σ bound a, 2-cell ϵ_2 which crosses itself along a non-singular segment $\alpha\beta$, α and β being vertices on σ. The non-singular image e_2, of ϵ_2, is in $(1, 1)$ correspondence with ϵ_2 except for two segments ab and $a'b'$, each of which corresponds to $\alpha\beta$. Let b and a' be inside e_2, and a and b' on s, the boundary of e_2 and the image of σ. Let x and x' be vertices on ab and $a'b'$ respectively, having the same image on $\alpha\beta$, and let t be any segment in e_2 joining x to x' and corresponding to a simple circuit τ, in M. If t does not meet ab or $a'b'$ except in x and x', we call it a *characteristic segment* and τ a *characteristic circuit* with respect to ϵ_2.

THEOREM 1. *If a given characteristic circuit is an elementary circuit, ϵ_2 is contained in a 3-element.*

Let a given characteristic circuit τ be contained in a 3-element. This 3-element can be deformed into one which contains τ and also the double segment $\alpha\beta$, and finally into a 3-element which contains ϵ_2, by the methods used in proving Lemma 1.

THEOREM 2. *If σ is an elementary circuit, a given characteristic circuit is either an elementary circuit or a one-sided circuit in a non-singular projective plane.*

In virtue of Lemma 2 it is sufficient to assume that σ is contained in a punctured sphere U, and we assume that no characteristic circuit is contained in a punctured sphere. If any characteristic circuit were contained in a punctured sphere, it would follow from Lemma 2 and Theorem 1 that any circuit on ϵ_2, treated as a complex in M, would be an elementary circuit. After a slight deformation of (U) we assume that no face of ϵ_2, nor edge of the double line $\alpha\beta$, lies on (U), and that each vertex of $(U).\epsilon_2$ is incident with precisely two edges of $(U).\epsilon_2$ if it does not lie on $\alpha\beta$, and with four edges if it lies on $\alpha\beta$. We also suppose that ϵ_2 does not touch (U), i.e. ϵ_2 crosses (U) at any common vertex.

Under these conditions the image of $(U).\epsilon_2$ in e_2 is a set of non-singular and non-intersecting circuits, which we shall call I. The vertices in which $\alpha\beta$ meets (U) will be called *double vertices*, and in the neighbourhood of a double vertex ξ, $(U).\epsilon_2$ corresponds to two segments in I meeting ab and $a'b'$ in x and x', the two images of ξ.

Our first step is to eliminate from $(U).\epsilon_2$ all the *isolated circuits*, that is to say, those which do not contain a double vertex. Such a circuit is the image of a circuit in I which does not meet ab or $a'b'$. If there is such a circuit, there is at least one which bounds a 2-element containing inside it no component of I. Let there be such a circuit, and let E_2 be the image on ϵ_2 of the 2-element in question. Let V stand for that one of the two regions U and $M-U$ which contains E_2. With a suitable covering, $N(E_2, V)$ is a 3-element which, with U, satisfies the conditions of Lemma 3. Therefore

$$U+N(E_2, V)$$

is either a punctured sphere containing U or a pair of punctured spheres, one of which contains σ. In either case U can be replaced by another punctured sphere containing σ such that the number of isolated circuits in $(U).\epsilon_2$ is reduced, and the number of double vertices is not increased. Without changing our notation we assume all the isolated circuits in $(U).\epsilon_2$ to have been eliminated in this way.

We proceed to replace the new region U by a punctured sphere such that no circuit in I meets ab or $a'b'$ more than once. No circuit in I separates b from a', since its image on (U) does not link σ. In the absence of isolated circuits, it follows that each circuit in I will then meet each of ab and $a'b'$ just once. If there is a circuit in I which meets either ab or $a'b'$ more than once, I say that there is at least one segment l, on some circuit in I, which, together with a segment of ab or $a'b'$, bounds a 2-element E_2^0, containing no other component of I or of ab or $a'b'$. Let C_1 be any circuit in I which meets ab or $a'b'$, say ab, more than once. The 2-element bounded by C_1 either contains both b and a' or neither. First assume that it contains neither. Then the segments of ab and $a'b'$ which are inside this 2-element have both ends on C_1, and at least one of these segments, together with an arc of C_1, bounds a 2-element C_2, containing no other component of ab or of $a'b'$. The existence of the required 2-element E_2^0, follows from a similar argument, applied to the segments of I lying in C_2.

If the 2-element bounded by C_1 contains both b and a', let x be the intersection of ab with C_1 which is nearest to b, and y the next nearest. Then the segment xy of ab, together with one of the two arcs into which C_1 is separated by x and y, bounds a 2-element containing neither b nor a'. The existence of the 2-element E_2^0 now follows from the argument used in the previous case, the 2-element bounded by the two segments xy playing the part of the original 2-element bounded by C_1.

Let E_2 and λ be the images of E_2^0 and l in M, and let V stand for whichever of U or $M-U$ contains E_2. Then the double vertices of $(U).\epsilon_2$ in which $\alpha\beta$ meets λ are eliminated by means of the deformation

$$U \to U+N(E_2, V),$$

and no new double vertices are introduced. Any isolated circuits which are created by this deformation can be eliminated as before without increasing the number of double vertices. It follows from induction on the latter that σ is contained in a punctured sphere U such that each circuit in I meets each of the segments ab and $a'b'$ just once.

Now let $c_1, c_2,..., c_k$ be the circuits in I, and let c_i meet ab in p_i and $a'b'$ in q_i. We suppose c_{i+1} to be contained in the 2-element bounded by c_i, so that p_{i+1} lies between p_i and b and q_{i+1} between q_i and a'. The vertices p and q occur in pairs, the vertices in each pair having the same image on $\alpha\beta$. From the order in which they occur it follows that

$$q_i = p'_{k-i+1}$$

where p_j and p'_j have the same image.

Since we are assuming that a characteristic circuit for ϵ_2 is not an elementary circuit, the set I is not empty and $k > 0$. Moreover k is even, say $k = 2r$, since α and β are both inside U. Let C_2 be one of the 2-elements into which the band bounded by c_r and c_{r+1} is separated by the segments $p_r p_{r+1}$ of ab and $q_r q_{r+1}$ of $a'b'$. Since

$$q_r = p'_{r+1}, \qquad q_{r+1} = p'_r,$$

the image of C_2 in M is a non-singular Möbius band μ_2, with its boundary (to modulus 2) on (U) and no other component on (U). A segment in C_2 joining a vertex on $p_r p_{r+1}$ to the corresponding vertex on $p'_r p'_{r+1}$ determines a characteristic circuit in M which is a one-sided circuit on μ_2. A 2-element on (U) bounded by (μ_2) can be added to μ_2 to provide a non-singular projective plane on which a characteristic circuit is a one-sided circuit.

From an argument similar to the one which led to the 2-cell E_2^0, bounded by l and a segment of ab or $a'b'$, it follows that any characteristic circuit can be transformed into any other by a non-singular deformation. Therefore the theorem follows from Lemma 1.

Corollary. *In a manifold with no torsion, σ is an elementary circuit, if and only if a given characteristic circuit is an elementary circuit.*

We conclude this section with a theorem which is not needed for the subsequent sections, but which may be of some general interest. Let $\sigma^1, \ldots, \sigma^n$ be any set of non-singular circuits in M, each of which bounds a 2-cell, and let each circuit σ^i contain at least one segment m^i, which does not belong to any of the others. Then the theorem is:

Theorem 3. *The circuits σ bound a set of 2-cells whose intersections with themselves and each other consist of non-singular double segments joining vertices on $\sigma^1, \ldots, \sigma^n$.*

Let $\sigma^i = (\epsilon_2^i)$. Each of the 2-cells ϵ can obviously be deformed into a 2-cell which has no segment in common with any of the circuits σ, which does not touch any of these circuits, and which does not intersect σ^i except in m^i. After further slight deformations it may be assumed that $\epsilon_2^1, \ldots, \epsilon_2^n$ intersect themselves and each other at most in a set of double segments, at which two sheets belonging to ϵ_2^i and ϵ_2^j cross ($i = j$; or $i \neq j$), and triple points at which three sheets cross.† The branch points may be eliminated by Dehn's method, or by the similar method of cutting used in T.M. for the reduction of the model R.

† Cf. M. Dehn, *Math. Annalen*, 69 (1910), 147.

Each double line corresponds to two lines on e_2^i and e_2^j respectively ($i = j$; or $i \neq j$), e_2^k being the non-singular image of ϵ_2^k. It is either a segment (possibly singular) or a circuit. Let α, on one of the arcs m, be an end-point of a double segment which contains a triple point. Taking α as the first point of the double segment, let β be the first triple point, and let the sheet which cuts the double segment at β be on ϵ_2^k. Then the triple point β is eliminated by the deformation

$$\epsilon_2^k \to \epsilon_2^k + (E_3),$$

where E_3 is a 3-element defined, as in the proof of Lemma 1, in terms of $N(\alpha\beta, M)$ and the neighbourhood of β on the sheet which cuts $\alpha\beta$. No new triple points are created, and induction shows that all the triple points on the double segments can be eliminated.

The theorem now follows from induction on the number of double circuits and an argument similar to the one used in the proof of Lemma 1 to eliminate the circuits in $S \,.\, E_2$.

Any double segment γ corresponds to segments g^i and g^j on e_2^i and e_2^j. If g^i has one end on (e_2^i) and one end inside e_2^i, the same will be true of g^j with respect to e_2^j. In this case γ will be described as of the first type. Otherwise either g^i or g^j, say g^i, will have both its ends on (e_2^i), and g^j will have both ends inside e_2^j. If g^i and part of the segment on (e_2^i) corresponding to m^i bound a 2-element on e_2^i containing no other double segment, g^i can be eliminated by a deformation similar to the second deformation used in proving Lemma 1. In any case a deformation of the first kind, applied to the sheet g^i of ϵ_2^i, replaces γ by two double segments of the first type. For a vertex of g^j can be joined to a vertex on the image of m^j by a segment which does not meet any other double segment, and the image of this segment on ϵ_2^j can take the place of t in the proof of Lemma 1. Thus all the double segments may be replaced by double segments of the first type.

3. By a *ring* we shall mean an (orientable) anchor ring composed of two 3-elements E_1 and E_2, meeting in a pair of 2-elements common to (E_1) and (E_2), these 2-elements having no common vertex. The boundary of either 2-element will be called a *meridian* circuit, and a circuit on the boundary of the ring which intersects a meridian in a single vertex will be called a *longitudinal* circuit, or a *longitude.* We shall also describe as a longitude any circuit in the ring which is isotopic to a longitudinal circuit on the boundary.

LEMMA 4. *Either of two symbolic rings with a common boundary can be transformed internally into the other provided they have a meridian circuit in common.*

Let R_1 and R_2 be the two rings and let

$$R_1 = E_1 + E_1^*,$$

where E_1 and E_1^* meet in 2-elements bounded by meridian circuits m and m^*, m being also a meridian of R_2. After a suitable internal subdivision $R_2 \to R_2'$, it is clear that

$$R_2' = E_2 + E_2^*,$$

where E_2 and E_2^* are 3-elements meeting in 2-elements bounded by m and m^*. Then R_1 and R_2' can be transformed internally into the same ring by starring first E_i and E_i^* ($i = 1, 2$) and then the common 2-elements. Since $R_2 \to R_2'$ internally, the lemma is established.

Let R_1 be a ring contained in a symbolic manifold M, and let R_2 be a symbolic ring whose boundary is identical with (R_1), the two rings having a common meridian. Then from Lemma 4 we have

LEMMA 5. *If R_2 has no internal component in common with M,*

$$M \to M - R_1 + R_2$$

by transformations which are internal to R_1.

A circuit σ in a ring M will be called a *self-linking circuit of the first type* if it bounds a 2-cell of the kind described in the last section, with a longitude as a characteristic circuit. If M is taken to be the region outside an unknotted tube m, in a 3-sphere, the diagram in § 2 represents a self-linking circuit of the first type. If $n > 1$, σ will be called a self-linking circuit of the nth type if it bounds a 2-cell of the kind described in § 2, with a self-linking circuit of the $(n-1)$th type as a characteristic circuit. From the corollary to Theorem 2 and induction on n it follows that a self-linking circuit of the nth type is self-linking in the sense of § 2.

Let s be a self-linking circuit of the first type inside a symbolic ring R. After a suitable subdivision, $N(s, R)$ will be a ring S, any longitude in which will be a self-linking circuit of the first type in R. Let m be a meridian circuit and l a longitude on (S), l and m having a single vertex in common. Let M and L respectively be meridian and longitudinal circuits on (R), meeting in a single vertex.

After a suitable subdivision† of R we may suppose (R) and (S) to

† i.e. a general subdivision (Newman, loc. cit.).

be congruent, L and M corresponding to l and m respectively. Without altering our notation we suppose this to be the case.

Let $$U = P(a_1, a_2, \ldots, a_N)$$
be the region $R-S$, and let
$$U^i = P(a_1^i, a_2^i, \ldots, a_N^i) \qquad (i = 1, 2, \ldots)$$
be an infinite sequence of copies of U, P being the same function in each case. Let
$$(R^i) = F(b_1^i, \ldots, b_k^i), \qquad (S^i) = F(c_1^i, \ldots, c_k^i),$$
the b's and c's being certain of the vertices a, and the congruence $(R) \equiv (S)$ being given by
$$b_\lambda \to c_\lambda.$$
Writing b_λ^0 for b_λ, let b_λ^{i-1} be substituted for c_λ^i in U^i ($\lambda = 1, \ldots, k$; $i = 1, 2, \ldots$), and let V^i be the region into which U^i is thus transformed. Then I say that the manifold
$$W = R + \sum_{i=1}^{\infty} V^i$$
satisfies the conditions

(i) *every circuit bounds a* 2-*cell*;

(ii) *every finite* 2-*cycle bounds a finite region*;

(iii) *every non-singular* 2-*sphere bounds a* 3-*element*;

and that W is not a formal 3-cell.

For let $$W^n = R + \sum_{i=1}^{n} V^i,$$
with $W^0 = R$. Then it follows from induction on n and Lemma 5 that W^n is a ring for every value of n, a meridian circuit on (W^n) $((W^n) = (R^n))$ corresponding to a meridian on (S^{n+1}) in the congruence
$$b_\lambda^n \to c_\lambda^{n+1}.$$
The conditions (ii) and (iii) are satisfied by any ring. Therefore they are satisfied by W, since any finite region in W is contained in W^n for some value of n.

Let l^n be the longitude on (S^n) which corresponds to l on (S). Then l^n bounds a 2-cell in W^{n+1}, since any longitude in S is deformable into s, and s bounds a 2-cell in R. Therefore any circuit in W^n bounds a 2-cell in W^{n+1}, and it follows that the manifold W satisfies the first, and therefore all the conditions (i), (ii), (iii).

Since $$R^n - S^n = U^n \to V^n = W^n - W^{n-1}$$
on replacing c_λ^n by b_λ^{n-1}, it follows from Lemma 5 that $R^n \to W^n$ by

a series of simple transformations which, except for the substitutions $(c_\lambda^n, b_\lambda^{n-1})$, are internal to S^n. It follows that R^n and W^n have a common subdivision which, except for the substitutions $(c_\lambda^n, b_\lambda^{n-1})$, leaves U^n and V^n unaltered.† Since l is a self-linking circuit of the first type in R, it follows that L^{n-1} is a self-linking circuit of the first type in W^n, L^i being the circuit on (R^i) corresponding to L on (R). Therefore any longitudinal circuit in W^{n-1} is a self-linking circuit of the first type in W^n. It follows from induction on k that any self-linking circuit of the kth type in W^{n-1} is a self-linking circuit of the $(k+1)$th type in W^n. From induction on p it follows that any self-linking circuit of the kth type in W^n is a self-linking circuit of the $(k+p)$th type in W^{n+p}. In particular, l is a self-linking circuit of the nth type in W^n.

If W were a formal 3-cell, some subdivision of l would be contained in an element which would itself be contained in a subdivision of W^n for some value of n. This would contradict the fact that l is a self-linking circuit in W^n. Therefore W is not a formal 3-cell.

4. Leaving aside the question whether or no all the spaces defined by the methods of § 3 are equivalent, we shall show that one of them has a semilinear map in a semilinear 3-sphere.‡

By an unknotted ring R, in a 3-sphere H, we shall mean one such that $H-R$ is a ring. With a suitable covering it is clear that the neighbourhood of an unknotted circuit is an unknotted ring.

Let R be an unknotted ring in a 3-sphere H, which we take to be a rectilinear simplicial complex in Euclidean N-space ($N > 3$). Let s be an unknotted, self-linking circuit of the first type in R (cf. the diagram in § 2), and let W be the manifold defined by means of R, s and the construction given in § 3. We shall take W to be a rectilinear complex in Hilbert space, with the symbol given in § 3. Since s is unknotted, $H-S$ is a ring, and we assume the congruence $(R) \equiv (S)$ of § 3 to be such that meridian circuits of R and $H-R$ correspond to meridians of S and $H-S$, respectively.

Let T^n be an unknotted ring in H, and let W^n be mapped semilinearly on $H-T^n$ in such a way that the image in the congruence $(R) \equiv (W^n)$ of a meridian on $H-R$ is mapped on a meridian of T^n. Then the initial congruence $(R) \equiv (S)$ determines a semilinear mapping $(S) \rightarrow (T^n)$, such that a meridian of $H-S$ corresponds to

† J. H. C. Whitehead, *Proc. Cambridge Phil. Soc.* loc. cit.

‡ Cf. J. H. C. Whitehead, *Proc. National Ac. of Sci.*, 21 (1935), 364–6.

a meridian of T^n. From Lemma 5 it follows that $H-S$ can be mapped semilinearly on T^n in such a way that the transformation $H-S \to T^n$ coincides with $(S) \to (T^n)$ on (S). Let T^{n+1} be the image of $H-R$ in $H-S \to T^n$. Then $T^n - T^{n+1}$ is the image of $R-S$, and $W^{n+1}-W^n$ is mapped semilinearly on $T^n - T^{n+1}$ in such a way that the transformations

$$W^n \to H-T^n, \qquad W^{n+1}-W^n \to T^n-T^{n+1}$$

can be united to form a semilinear transformation

$$W^{n+1} \to H-T^{n+1}.$$

Moreover a meridian of $H-R$, and therefore the corresponding circuit of W^{n+1}, is mapped on a meridian of T^{n+1}.

If $T^0 = H-R$, an infinite sequence of rings T^0, T^1,... ($T^n \subset T^{n+1}$) is defined inductively, and W is mapped semilinearly on $H-X$, where X is the set of points common to T^0, T^1,... .