# RADON-NIKODYM THEOREM, HAHN-JORDAN DECOMPOSITION AND LEBESGUE DECOMPOSITION

## 1. Introduction

This notes are devoted to advanced notions in measure theory. Tools presented here are indispensable in probability theory, statistics and applications to geometry. We refer to [Monygham, 2018] for basic measure theory and to [Monygham, 2019] for integration theory.

## 2. Signed and Complex Measures

In this section we define extensions of the notion of measure.

**Definition 2.1.** Let $(X, \Sigma)$ be a measurable space. *A signed measure on* $(X, \Sigma)$ is a function $\nu : \Sigma \to \overline{\mathbb{R}}$ such that $\nu(\varnothing) = 0$ and

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of $\Sigma$. We also say that $\nu$ is *a real measure on* $(X, \Sigma)$ if it is signed measure and takes values in $\mathbb{R}$.

**Definition 2.2.** Let $(X, \Sigma)$ be a measurable space. *A complex measure* is a function $\nu : \Sigma \to \mathbb{C}$ such that $\nu(\varnothing) = 0$ and

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n)$$

for every family $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets of $\Sigma$.

**Definition 2.3.** Let $(X, \Sigma)$ be a measurable space and let $\mu$ be a measure on $(X, \Sigma)$. Assume that $\nu$ is either complex or signed measure on $(X, \Sigma)$. Suppose that for every set $A$ in $\Sigma$ we have

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

Then we write $\nu \ll \mu$ and say that $\nu$ is *absolutely continuous with respect to* $\mu$.

**Definition 2.4.** Let $(X, \Sigma)$ be a measurable space and let $\mu$, $\nu$ be two measures either complex or signed on $(X, \Sigma)$. Suppose that there exists a set $S \in \Sigma$ such that

$$\mu(A \cap S) = 0,\ \nu(A \setminus S) = 0$$

for every $A \in \Sigma$. Then we write $\nu \perp \mu$ and say that $\nu$ is *singular with respect to* $\mu$.

## 3. Hahn-Jordan Decomposition

**Theorem 3.1** (Hahn-Jordan decomposition)**.** *Let $(X, \Sigma)$ be a measurable space and $\nu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Then there exists the unique pair of measures $\nu_+, \nu_- : \Sigma \to [0, +\infty]$ such that*

$$\nu = \nu_+ - \nu_-$$

*and $\nu_+ \perp \nu_-$.*

For the proof we need the following notion.

**Definition 3.2.** Let $(X, \Sigma, \nu)$ be a space with signed measure. A set $A \in \Sigma$ is *positive* if for every subset $B$ of $A$ such that $B \in \Sigma$ we have inequality $\nu(B) \geq 0$.

**Lemma 3.2.1.** *Let $B \in \Sigma$ be a set such that $\nu(B) \in \mathbb{R}$ and $\nu(B) > 0$. Then there exists a positive set $C \subseteq B$ such that $\nu(B) \leq \nu(C)$.*

*Proof of the lemma.* First note that all sets $A \in \Sigma$ contained in $B$ have finite measure (we left the proof as an exercise for the reader). For every subset $A \in \Sigma$ contained in $B$ we define

$$\delta(A) = \max\left\{\frac{1}{2}\inf\big\{\nu(D)\,\big|\,D \text{ is a subset of } A \text{ in } \Sigma\big\}, -1\right\}$$

Note that $\delta(A) \leq 0$. Now we define a sequence $\{D_n\}_{n\in\mathbb{N}}$ of disjoint subsets of $B$ and members of $\Sigma$. This is done recursively as follows. If $D_0, ..., D_n$ are defined, then we pick $D_{n+1}$ as a subset of $B \setminus (D_0 \cup ... \cup D_n)$ in $\Sigma$ such that

$$\nu(D_{n+1}) \leq \delta\big(B \setminus (D_0 \cup ... \cup D_n)\big)$$

Let

$$C = B \setminus \bigcup_{n\in\mathbb{N}} D_n$$

be a subset of $B$. Clearly $C \in \Sigma$ and for every $n \in \mathbb{N}$ we have

$$\delta(C) \geq \delta\big(B \setminus (D_0 \cup ... \cup D_n)\big)$$

Thus

$$\nu(C) = \nu(B) - \sum_{n\in\mathbb{N}} \nu(D_n) \geq \nu(B) - \sum_{n\in\mathbb{N}} \delta\big(B \setminus (D_0 \cup ... \cup D_n)\big) = \nu(B) - \sum_{n\in\mathbb{N}} \delta(C)$$

Since $\nu(C) \in \mathbb{R}$, we derive that $\delta(C) = 0$. This implies that $C$ is a positive set and $\nu(C) \geq \nu(B)$. □

*Proof of the theorem.* Assume that for every $A \in \Sigma$ we have $\nu(A) \in \mathbb{R} \cup \{-\infty\}$. Now consider

$$\alpha = \sup\big\{\nu(C)\,\big|\,C \text{ is positive}\big\}$$

We can find a nondecreasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of nonnegative real numbers that converges to $\alpha$ and such that for every $n \in \mathbb{N}$ there exists a positive set $C_n$ with $\nu(C_n) = \alpha_n$. Now pick $P = \bigcup_{n\in\mathbb{N}} C_n$. Obviously $P$ is positive and $\nu(P) = \alpha$. In particular, $\alpha \in \mathbb{R}$. Assume that there exists $B \in \Sigma$ such that $B \subseteq X \setminus P$ and $\nu(B) > 0$. According to Lemma 3.2.1 we deduce that there exists a positive set $C$ inside $B$ such that $\nu(B) \leq \nu(C)$. Then we get

$$\alpha = \nu(P) < \nu(P) + \nu(C) = \nu(P \cup C)$$

and $P \cup C$ is positive. This contradicts the definition of $\alpha$. Hence for every $B \subseteq X \setminus P$ such that $B \in \Sigma$ we have $\nu(B) \leq 0$. Thus measures

$$\nu_+(A) = \nu(A \cap P),\ \nu_-(A) = -\nu(A \setminus P)$$

defined for $A \in \Sigma$ satisfy the assertion of the theorem. This finishes the proof of the Hahn-Jordan decomposition under the assumption that $\nu(A) \in \mathbb{R} \cup \{-\infty\}$ for all $A \in \Sigma$.
If $\nu(A) \in \mathbb{R} \cup \{+\infty\}$ for every $A \in \Sigma$, then we apply the result above for $-\nu$. Finally the case $\nu(A_1) = -\infty$ and $\nu(A_2) = +\infty$ for some $A_1, A_2 \in \Sigma$ yields to the contradiction. Hence Hahn-Jordan decomposition is proved. □

**Corollary 3.3.** *Let $(X, \Sigma)$ be a measurable space and $\nu : \Sigma \to \overline{\mathbb{R}}$ be a signed measure. Then either $\nu_+$ or $\nu_-$ is finite.*

*Proof.* According to Theorem 3.1 we have $\nu = \nu_+ - \nu_-$ and both $\nu_+$, $\nu_-$ are measures such that $\nu_+ \perp \nu_-$. We cannot have $\nu_+(X) = \nu_-(X) = +\infty$, because then $\nu(X)$ would be undefined. This implies that either $\nu_+(X) \in \mathbb{R}$ or $\nu_-(X) \in \mathbb{R}$. □

## 4. Lebesgue decomposition

**Definition 4.1.** Let $(X,\Sigma)$ be a measurable space and $\mu:\Sigma\to\overline{\mathbb{R}}$ be a signed measure. We say that $\mu$ is $\sigma$-finite if there exists a decomposition

$$X=\bigcup_{n\in\mathbb{N}}X_n$$

onto pairwise disjoint elements of $\Sigma$ such that $\mu(X_n)\in\mathbb{R}$ for every $n\in\mathbb{N}$.

**Theorem 4.2** (Lebesgue decomposition)**.** *Let $(X,\Sigma)$ be a measurable space and let $\mu$ be a measure on $(X,\Sigma)$. Suppose that $\nu$ is either a signed and $\sigma$-finite measure or a complex measure on $(X,\Sigma)$. Then there exists a unique decomposition*

$$\nu=\nu_s+\nu_a$$

*of measure $\nu$ such that $\nu_s\perp\mu$ and $\nu_a\ll\mu$.*

*Proof.* Suppose first that $\nu$ is a finite measure. Consider

$$\alpha=\sup_{A\in\Sigma,\,\mu(A)=0}\nu(A)$$

Since $\nu$ is finite, we derive that $\alpha\in\mathbb{R}$. Consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n\in\Sigma$, $\mu(A_n)=0$ for every $n\in\mathbb{N}$ and $\lim_{n\to+\infty}\nu(A_n)=\alpha$. Define $S=\bigcup_{n\in\mathbb{N}}A_n$. Then $\mu(S)=0$ and $\nu(S)=\alpha$. Moreover, if $A\in\Sigma$ and $A\cap S=\varnothing$, then $\mu(A)=0$ implies $\nu(A)=0$. Now we define $\nu_s(A)=\nu(A\cap S)$ and $\nu_a(A)=\nu(A\setminus S)$ for every $A\in\Sigma$. Then $\nu=\nu_s+\nu_a$ and $\nu_s\perp\mu$, $\nu_a\ll\mu$.
Now assume that $\nu$ is $\sigma$-finite measure. There exists a decomposition

$$X=\bigcup_{n\in\mathbb{N}}X_n$$

onto pairwise disjoint elements of $\Sigma$ such that $\nu(X_n)\in\mathbb{R}$ for every $n\in\mathbb{N}$. We define $\nu_n(A)=\nu(A\cap X_n)$ for each $n\in\mathbb{N}$ and $A\in\Sigma$. Then $\nu_n$ is a finite measure. By the case above we find $\nu_n=\nu_{ns}+\nu_{na}$ and $\nu_{ns}\perp\mu$, $\nu_{na}\ll\mu$ for some measures on $\Sigma$. Now we define

$$\nu_s=\sum_{n\in\mathbb{N}}\nu_{ns},\ \nu_a=\sum_{n\in\mathbb{N}}\nu_{an}$$

Then $\nu=\nu_s+\nu_a$ and $\nu_s\perp\mu$, $\nu_a\ll\mu$.
Now consider the case when $\nu$ is $\sigma$-finite and signed measure. According to Theorem 3.1 we write $\nu=\nu_+-\nu_-$ for measures $\nu_+,\nu_-$ such that $\nu_+\perp\nu_-$. Then $\nu_+,\nu_-$ are $\sigma$-finite measures. According to previous case we can write $\nu_+=\nu_{+s}+\nu_{+a}$, $\nu_-=\nu_{-s}+\nu_{-a}$ for measures such that $\nu_{+s}\perp\mu,\nu_{-s}\perp\mu,\nu_{+a}\ll\mu,\nu_{-a}\ll\mu$. Then $\nu_s=\nu_{+s}-\nu_{-s},\nu_a=\nu_{+a}-\nu_{-a}$ are signed measures and $\nu_s\perp\mu,\nu_a\ll\mu$.
Finally assume that $\nu$ is complex. Then $\nu=\nu^r+i\cdot\nu^i$, where $\nu^r$ and $\nu^i$ are finite, signed measures. Form the case above we have decompositions

$$\nu^r=\nu^r_s+\nu^r_a,\ \nu^i=\nu^i_s+\nu^i_s$$

and $\nu^r_s\perp\mu$, $\nu^i_s\perp\mu$, $\nu^r_a\ll\mu$, $\nu^i_a\ll\mu$. Then complex measures

$$\nu_s=\nu^r_s+i\cdot\nu^i_s,\ \nu_a=\nu^r_a+i\cdot\nu^i_a$$

satisfy $\nu_s\perp\mu$, $\nu_a\ll\mu$. □

## 5. Radon-Nikodym theorem

In this section we prove famous result of Radon and Nikodym.

**Theorem 5.1** (Radon-Nikodym)**.** *Let $(X,\Sigma)$ be a measurable space and let $\mu$ be a $\sigma$-finite measure on $(X,\Sigma)$. Suppose that $\nu\ll\mu$ for $\nu$ that is either complex measure or $\sigma$-finite, signed measure. Then there exists a measurable function $f:X\to\mathbb{C}$ such that*

$$\nu(A)=\int_A f\,d\mu$$

*for every $A\in\Sigma$.*

*Proof for finite measures $\mu, \nu$.* Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}$. According to Theorem 3.1 there exists a set $P_{n,k} \in \Sigma$ such that

$$\left(\nu - \frac{k}{2^n} \cdot \mu\right)\left(A \cap P_{n,k}\right) \geq 0, \left(\nu - \frac{k}{2^n} \cdot \mu\right)\left(A \setminus P_{n,k}\right) \leq 0$$

for every $A \in \Sigma$. We may assume that $P_{n,0} = X$, $P_{n,k+1} \subseteq P_{n,k}$ and $P_{n,k} = P_{n+1,2k}$ for every $n, k \in \mathbb{N}$. Since $\nu \ll \mu$ and $\nu$ is finite, we derive that

$$\mu\left(\bigcap_{k\in\mathbb{N}} P_{n,k}\right) = \nu\left(\bigcap_{k\in\mathbb{N}} P_{n,k}\right) = 0$$

and may assume that this set is empty for every $n \in \mathbb{N}$. Pick $n \in \mathbb{N}$. We define a function $f_n : X \to \mathbb{C}$ by formula

$$f_n(x) = \sum_{k\in\mathbb{N}} \frac{k}{2^n} \cdot \mathbb{1}_{P_{n,k}\setminus P_{n,k+1}}(x)$$

Clearly $f_n$ is a measurable, nonnegative function. If $m, n \in \mathbb{N}$ and $n \leq m$, then we have

$$f_n(x) \leq f_m(x) \leq f_n(x) + \frac{1}{2^n}$$

Thus $\{f_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence of measurable functions convergent uniformly to a measurable function $f : X \to \mathbb{C}$. Moreover, for every $A \in \Sigma$ and $n \in \mathbb{N}$ we have

$$\nu(A) - \frac{1}{2^n}\mu(A) = \sum_{k\in\mathbb{N}} \nu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) - \frac{1}{2^n}\mu(A) \leq$$

$$\leq \sum_{k\in\mathbb{N}} \frac{k+1}{2^n}\mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) - \frac{1}{2^n}\sum_{k\in\mathbb{N}} \mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) \leq$$

$$\leq \sum_{k\in\mathbb{N}} \frac{k}{2^n}\mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) \leq \sum_{k\in\mathbb{N}} \nu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right) = \nu(A)$$

and since

$$\int_A f_n \, d\mu = \sum_{k\in\mathbb{N}} \frac{k}{2^n}\mu\left(A \cap \left(P_{n,k} \setminus P_{n,k+1}\right)\right)$$

we derive that

$$\nu(A) - \frac{1}{2^n}\mu(A) \leq \int_A f_n \, d\mu \leq \nu(A)$$

This inequality together with monotone convergence theorem imply that

$$\nu(A) = \lim_{n\to+\infty} \int_A f_n \, d\mu = \int_A f \, d\mu$$

This finishes the proof for finite measures $\nu, \mu$. □

*Reduction to finite case.* Assume now that $\nu$ and $\mu$ are $\sigma$-finite measures on $(X, \Sigma)$. Then there exists a decomposition

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

onto disjoint subsets in $\Sigma$ such that $\nu(X_n) \in \mathbb{R}$ and $\mu(X_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we define $\nu_n(A) = \nu(A \cap X_n)$ and $\mu_n(A) = \mu(A \cap X_n)$ for $A \in \Sigma$. Since $\nu \ll \mu$, we derive that $\nu_n \ll \mu_n$ for every $n \in \mathbb{N}$. Measures $\{\nu_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ are finite. By finite case of the theorem we deduce that for each $n \in \mathbb{N}$ there exists a measurable function $f_n : X \to \mathbb{C}$ such that

$$\nu_n(A) = \int_A f_n \, d\mu_n$$

for every $A \in \Sigma$. Note that $f_n$ has nonnegative values $\mu$-almost everywhere and can be set equal to zero outside $X_n$. Thus

$$\nu_n(A) = \int_A f_n \, d\mu_n = \int_A f_n \, d\mu$$

for every $A \in \Sigma$. Therefore, we deduce that

$$\nu(A) = \sum_{n\in\mathbb{N}} \nu(A \cap X_n) = \sum_{n\in\mathbb{N}} \nu_n(A) = \sum_{n\in\mathbb{N}} \int_A f_n \, d\mu = \int_A \left( \sum_{n\in\mathbb{N}} f_n \right) d\mu$$

by monotone convergence theorem.
Assume now that both $\nu$ is $\sigma$-finite, signed measure. In this situation we may write $\nu = \nu_+ - \nu_-$ for measures $\nu_+, \nu_-$ such that $\nu_+ \perp \nu_-$. Then $\nu_+ \ll \mu$ and $\nu_- \ll \mu$. There exists a set $P \in \Sigma$ such that $\nu_-(P) = \nu_+(X \setminus P) = 0$. By the case considered previously there exist measurable functions $f_+ : X \to \mathbb{C}$, $f_- : X \to \mathbb{C}$ such that

$$\nu_+(A) = \int_A f_+ \, d\mu, \; \nu_-(A) = \int_A f_- \, d\mu$$

for every $A \in \Sigma$. Moreover, we may assume that $f_+$ is equal to zero outside $P$ and $f_-$ is equal to zero outside $X \setminus P$. From this we have

$$\nu(A) = \nu_+(A) + \nu_-(A) = \int_A f_+ \, d\mu + \int_A f_- \, d\mu = \int_A (f_+ - f_-) \, d\mu$$

for every $A \in \Sigma$.
Suppose that $\nu$ is complex measure. Write $\nu = \nu_r - i \cdot \nu_i$. Then both $\nu_r, \nu_-$ are finite, signed measures. Moreover, we have $\nu_r \ll \mu, \nu_i \ll \mu$. There exist measurable functions $f_r : X \to \mathbb{C}$ and $f_i : X \to \mathbb{C}$ that are real valued and satisfy

$$\nu_r(A) = \int_A f_r \, d\mu, \; \nu_i(A) = \int_A f_i \, d\mu$$

for every $A \in \Sigma$. Thus

$$\nu(A) = \nu_r(A) + i \cdot \nu_i(A) = \int_A f_r \, d\mu + i \cdot \int_A f_i \, d\mu = \int_A (f_r + i \cdot f_i) \, d\mu$$

for every $A \in \Sigma$. □

## 6. Banach space of complex measures

**Proposition 6.1.** *Let $\mu$ be a complex measure on a measurable space $(X, \Sigma)$. For every $A \in \Sigma$ we define*

$$|\mu|(A) = \sup \left\{ \sum_{n\in\mathbb{N}} |\mu(A_n)| \,\middle|\, A = \bigcup_{n\in\mathbb{N}} A_n \text{ is a partition of } A \text{ onto subsets in } \Sigma \right\}$$

*Then $|\mu|$ is a finite measure on $(X, \Sigma)$.*

*Proof.* Let $\mu = \mu^r + i \cdot \mu^i$ be decomposition onto real and imaginary part. Then $\mu^r, \mu^i$ are finite, signed measures. By Theorem 3.1 we derive that there exist decompositions $\mu^r = \mu^r_+ - \mu^r_-$, $\mu^i = \mu^i_+ - \mu^i_-$ such that $\mu^r_+, \mu^r_-, \mu^i_+, \mu^i_-$ are finite measures and $\mu^r_+ \perp \mu^r_-, \mu^i_+ \perp \mu^i_-$. Then for every partition

$$A = \bigcup_{n\in\mathbb{N}} A_n$$

of $A \in \Sigma$ onto sets in $\Sigma$ we have

$$\sum_{n\in\mathbb{N}} |\mu(A_n)| = \sum_{n\in\mathbb{N}} \sqrt{(\mu^r(A_n))^2 + (\mu^i(A_n))^2} \leq \sum_{n\in\mathbb{N}} \left( |\mu^r(A_n)| + |\mu^i(A_n)| \right) \leq$$

$$\leq \sum_{n\in\mathbb{N}} \left( \mu^r_+(A_n) + \mu^r_-(A_n) + \mu^i_+(A_n) + \mu^i_-(A_n) \right) = \mu^r_+(A) + \mu^r_-(A) + \mu^i_+(A) + \mu^i_-(A)$$

Right hand side of the inequality does not depend on the partition and hence

$$|\mu|(A) \leq \mu^r_+(A) + \mu^r_-(A) + \mu^i_+(A) + \mu^i_-(A)$$

This implies that $|\mu|(A) \in \mathbb{R}$ for every $A \in \Sigma$. Note also that $|\mu|(\varnothing) = 0$. Suppose now that $A \in \Sigma$ and we have partitions

$$A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} C_n, \; A_n = \bigcup_{m\in\mathbb{N}} A_{n,m} \text{ for every } n \in \mathbb{N}$$

onto subsets in $\Sigma$. Then

$$\sum_{n\in\mathbb{N}} |\mu(C_n)| \leq \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |\mu(A_n \cap C_m)| \leq \sum_{n\in\mathbb{N}} |\mu|(A_n)$$

and

$$\sum_{n\in\mathbb{N}} \left( \sum_{m\in\mathbb{N}} |\mu(A_{n,m})| \right) \leq |\mu|(A)$$

These inequalities imply that

$$|\mu|(A) \leq \sum_{n\in\mathbb{N}} |\mu|(A_n) \leq |\mu|(A)$$

Therefore, $|\mu|$ is a finite measure. □

**Definition 6.2.** Let $\mu$ be a complex measure on $(X, \Sigma)$. Then we define

$$||\mu|| = |\mu|(X)$$

and call it *the total variation of* $\mu$.

**Theorem 6.3.** *Let $(X, \Sigma)$ be a measurable space and $\mathcal{M}(X, \Sigma)$ be a set of all complex measures on $(X, \Sigma)$. Then the following assertions hold.*

**(1)** *$\mathcal{M}(X, \Sigma)$ is a $\mathbb{C}$-linear space.*

**(2)** *The mapping*

$$\mathcal{M}(X, \Sigma) \ni \mu \mapsto ||\mu|| \in [0, +\infty)$$

*is a norm.*

**(3)** *Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of complex measures on $(X, \Sigma)$ that is a Cauchy sequence with respect to total variation. Then there exists a complex measure $\mu$ such that*

$$\lim_{n\to+\infty} \mu_n = \mu$$

*Moreover, for every $A \in \Sigma$ we have*

$$\lim_{n\to+\infty} \mu_n(A) = \mu(A)$$

*Proof.* We left **(1)** and **(2)** for the reader as an exercise. Fix $A \in \Sigma$. Then

$$|\mu_n(A) - \mu_m(A)| \leq |\mu_n - \mu_m|(A) \leq ||\mu_n - \mu_m||$$

for every $n, m \in \mathbb{N}$. Since $\{\mu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to total variation, we derive that there exists the limit $\mu(A)$ of $\{\mu_n(A)\}_{n\in\mathbb{N}}$. Suppose that

$$A = \bigcup_{k\in\mathbb{N}} A_k$$

for $A \in \Sigma$ and $A_k \in \Sigma$ for $k \in \mathbb{N}$. Assume that sets $\{A_k\}_{k\in\mathbb{N}}$ are disjoint. Pick $N \in \mathbb{N}$. Then

$$\sum_{k=0}^{N} |\mu_n(A_k) - \mu(A_k)| = \lim_{m\to+\infty} \sum_{k=0}^{N} |\mu_n(A_k) - \mu_m(A_k)| \leq$$

$$\leq \limsup_{m\to+\infty} \sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu_m(A_k)| \leq \limsup_{m\to+\infty} |\mu_n - \mu_m|(A) = \limsup_{m\to+\infty} ||\mu_n - \mu_m||$$

This implies that

$$\sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq \limsup_{m\to+\infty} ||\mu_n - \mu_m||$$

regardless of set $A$ and partition $\{A_k\}_{k\in\mathbb{N}}$. Thus we deduce that there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, convergent to zero such that

$$\sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq a_n$$

for every $n \in \mathbb{N}$, $A \in \Sigma$ and partition $\{A_k\}_{k\in\mathbb{N}}$ as above. Therefore, for fixed $N \in \mathbb{N}$ we have

$$\left|\mu(A) - \sum_{k=0}^{N} \mu(A_k)\right| \leq |\mu(A) - \mu_n(A)| + \left|\mu_n(A) - \sum_{k=0}^{N} \mu_n(A_k)\right| + \sum_{k=0}^{N} |\mu_n(A_k) - \mu(A_k)| \leq$$

$$\leq |\mu(A) - \mu_n(A)| + \left|\mu_n(A) - \sum_{k=0}^{N} \mu_n(A_k)\right| + \sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq 2a_n + \left|\mu_n(A) - \sum_{k=0}^{N} \mu_n(A_k)\right|$$

Hence we derive that

$$\mu(A) = \sum_{k\in\mathbb{N}} \mu(A_k)$$

thus $\mu$ is a complex measure and according to

$$\sum_{k\in\mathbb{N}} |\mu_n(A_k) - \mu(A_k)| \leq a_n$$

for every $n \in \mathbb{N}$ we deduce that

$$\lim_{n\to+\infty} |\mu_n - \mu|(A) = 0$$

for every $A \in \Sigma$. Hence also $\lim_{n\to+\infty} ||\mu_n - \mu|| = 0$. This finishes the proof of **(3)**. □

## 7. Applications of Radon-Nikodyn theorem

**Proposition 7.1.** *Let $\mu$ be a measure on a measurable space $(X, \Sigma)$ and $f : X \to \overline{\mathbb{R}}$ be a measurable, nonnegative function. We define*

$$\nu(A) = \int_A f \, d\mu$$

*for every $A \in \Sigma$. Then $\nu$ is a measure on $(X, \Sigma)$ and the equality*

$$\int_X g \, d\nu = \int_X g \cdot f \, d\mu$$

*holds if $g$ is either $\mu$-integrable function $g : X \to \mathbb{C}$ of a measurable, nonnegative function $g : X \to \overline{\mathbb{R}}$.*

*Proof.* Suppose that $A = \bigcup_{n\in\mathbb{N}} A_n$ for $A \in \Sigma$ and $A_n \in \Sigma$ for every $n \in \mathbb{N}$. Assume also that $\{A_n\}_{n\in\mathbb{N}}$ are pairwise disjoint. Then by monotone convergence theorem

$$\nu(A) = \int_A f \, d\mu = \int_X \mathbb{1}_A \cdot f \, d\mu = \int_X \left(\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} \cdot f\right) d\mu = \sum_{n\in\mathbb{N}} \int_X \mathbb{1}_{A_n} \cdot f \, d\mu = \sum_{n\in\mathbb{N}} \int_{A_n} f \, d\mu = \sum_{n\in\mathbb{N}} \nu(A_n)$$

Moreover, we have $\nu(\varnothing) = 0$. Thus $\nu$ is a measure on $(X, \Sigma)$.
For the second part of the statement note that the family of measurable, nonnegative functions $g : X \to \overline{\mathbb{R}}$ satisfying equality

$$\int_X g \, d\nu = \int_X g \cdot f \, d\mu$$

contains $\{\mathbb{1}_A\}_{A\in\Sigma}$, is closed under linear combinationss with nonnegative coefficients, if it contains nondecreasing sequence $\{g_n : X \to \overline{\mathbb{R}}\}_{n\in\mathbb{N}}$, then it also contains its pointwise limit. Thus this family contains all measurable, nonnegative functions $g : X \to \overline{\mathbb{R}}$. Since every real valued, $\nu$-integrable function $g : X \to \mathbb{C}$ is a difference of a two $\nu$-integrable, nonnegative functions, we deduce that this family contains all real, $\nu$-integrable functions. Finally, if it contains two $\nu$-integrable, real valued functions, then it contains its $\mathbb{C}$-linear combination. Thus it contains all $\nu$-integrable functions. □

**Theorem 7.2.** *Let $\mu$ be a complex measure on a measurable space $(X, \Sigma)$. There exists an $|\mu|$-integrable function $f : X \to \mathbb{C}$ such that*

$$\mu(A) = \int_A f \, d|\mu|$$

*for every $A \in \Sigma$ and $|f(x)| = 1$ for every $x$ in $X$.*

For the proof we need the following result.

**Lemma 7.2.1.** *Let $\mu$ be a measure on $(X,\Sigma)$. Suppose that $f : X \to \mathbb{C}$ is a measurable function and $F$ is a closed subset of $\mathbb{C}$. Assume that for every $A \in \Sigma$ such that $\mu(A) > 0$, we have*

$$\frac{1}{\mu(A)} \int_A f \, d\mu \in F$$

*Then $\mu\left(X \setminus f^{-1}(F)\right) = 0$.*

*Proof of the lemma.* Let $D$ be a closed disc in $\mathbb{C}$ such that $D \cap F = \varnothing$. If $\mu\left(f^{-1}(D)\right) > 0$, then

$$\frac{1}{\mu\left(f^{-1}(D)\right)} \int_{f^{-1}(D)} f \, d\mu \in D$$

by convexity of $D$. This implies that for every closed disc in $\mathbb{C}$ disjoint from $F$ we have $\mu\left(f^{-1}(D)\right) = 0$. Since $\mathbb{C} \setminus F$ can be covered by such discs, we derive that $\mu\left(X \setminus f^{-1}(F)\right) = 0$. □

*Proof of the theorem.* Consider Radon-Nikodym derivative $f : X \to \mathbb{C}$ of $\mu$ with respect to $|\mu|$. It exists according to Theorem 5.1 and is $|\mu|$-integrable because $\mu$ is complex measure. Since

$$\frac{1}{\mu(A)} \left| \int_A f \, d|\mu| \right| \leq \frac{1}{\mu(A)} \int_A |f| \, d|\mu| = \frac{|\mu|(A)}{\mu(A)} \leq 1$$

for every $A \in \Sigma$ such that $A \in \Sigma$, we derive by Lemma 7.2.1 that $f(x) \in D$ almost everywhere with respect to measure $|\mu|$, where $D$ is a closed unit disc in $\mathbb{C}$. Changing values of $f$ on a set of measure $|\mu|$ equal to zero, we may assume that $f(x) \in D$ for every $x$ in $X$.
Suppose next that $0 < \alpha < 1$ and denote $A_\alpha = f^{-1}\left(\{z \in \mathbb{C} \mid |f(z)| \leq \alpha\}\right)$. Let

$$A_\alpha = \bigcup_{n \in \mathbb{N}} A_n$$

be a decomposition on disjoint subsets in $\Sigma$. Then

$$\sum_{n \in \mathbb{N}} |\mu(A_n)| = \sum_{n \in \mathbb{N}} \left| \int_{A_n} f \, d|\mu| \right| \leq \sum_{n \in \mathbb{N}} \int_{A_n} |f| \, d|\mu| \leq \alpha \cdot \sum_{n \in \mathbb{N}} |\mu|(A_n) = \alpha \cdot |\mu|(A_\alpha)$$

Hence

$$|\mu|(A_\alpha) \leq \alpha \cdot |\mu|(A_\alpha)$$

Therefore, $|\mu|(A_\alpha) = 0$. Since $\alpha$ is arbitrary number in $(0,1)$, we deduce that

$$|\mu|\left(\left\{z \in \mathbb{C} \,\middle|\, |f(z)| < 1\right\}\right) = 0$$

Thus changing values of $f$ on a set of measure $|\mu|$ equal to zero, we may assume that $|f(x)| = 1$ for every $x$ in $X$. □

**Corollary 7.3.** *Let $\mu$ be a measure on a measurable space $(X,\Sigma)$ and $f : X \to \mathbb{C}$ be a $\mu$-integrable function. Define*

$$\nu(A) = \int_A f \, d\mu$$

*for every $A \in \mathbb{C}$. Then $\nu$ is a complex measure on $(X,\Sigma)$ and*

$$|\nu|(A) = \int_A |f| \, d\mu$$

*for every $A \in \Sigma$.*

*Proof.* Clearly $\nu(A) \in \mathbb{C}$ for every $A \in \Sigma$. Suppose that $A = \bigcup_{n \in \mathbb{N}} A_n$ for $A \in \Sigma$ and $A_n \in \Sigma$ for every $n \in \mathbb{N}$. Assume also that $\{A_n\}_{n \in \mathbb{N}}$ are pairwise disjoint. Then by dominated convergence theorem

$$\nu(A) = \int_A f \, d\mu = \int_X \mathbb{1}_A \cdot f \, d\mu = \int_X \left( \sum_{n \in \mathbb{N}} \mathbb{1}_{A_n} \cdot f \right) d\mu = \sum_{n \in \mathbb{N}} \int_X \mathbb{1}_{A_n} \cdot f \, d\mu = \sum_{n \in \mathbb{N}} \int_{A_n} f \, d\mu = \sum_{n \in \mathbb{N}} \nu(A_n)$$

Moreover, we have $\nu(\varnothing) = 0$. Thus $\nu$ is a complex measure on $(X, \Sigma)$. Since $f$ is $\mu$-integrable, there exists a $\sigma$-finite subset $\Omega \in \Sigma$ such that $|f(x)| = 0$ for $x \notin \Omega$. We define $\tilde{\mu}(A) = \mu(A \cap \Omega)$ for every $A \in \Sigma$. Clearly

$$\nu(A) = \int_A f \, d\mu = \int_A f \, d\tilde{\mu}$$

for every $A \in \Sigma$. Hence we have $|\nu| \ll \tilde{\mu}$ by definition of $\nu$ and $|\nu|$. Note that $\tilde{\mu}$ is a $\sigma$-finite measure. By Theorem 5.1 there exists a measurable function $g : X \to \mathbb{C}$ equal to zero outside $\Omega$ such that

$$|\nu|(A) = \int_A g \, d\tilde{\mu} = \int_A g \, d\mu$$

for every $A \in \Sigma$. We may assume that $g$ takes only nonnegative values. By Theorem 7.2 there exists a function $h : X \to \mathbb{C}$ such that

$$\nu(A) = \int_A h \, d|\nu|$$

for every $A \in \Sigma$ and $|h(x)| = 1$ for all $x$ in $X$. By Proposition 7.1 we deduce that

$$\int_A f \, d\mu = \nu(A) = \int_A h \, d|\nu| = \int_A h \cdot g \, d\mu$$

for every $A \in \Sigma$. Therefore, $f = h \cdot g$ almost everywhere with respect to $\mu$. Thus

$$g(x) = |h(x)| \cdot g(x) = |f(x)|$$

almost everywhere with respect to $\mu$. □

**Corollary 7.4.** *Let $(X, \Sigma)$ be a measurable space and $\mu$ be a measure on $\Sigma$. Then the map*

$$L^1(X, \mu) \ni f \mapsto \left( \Sigma \ni A \mapsto \int_A f \, d\mu \in \mathbb{C} \right) \in \mathcal{M}(X, \Sigma)$$

*is a $\mathbb{C}$-linear isometrical embedding of Banach spaces. If in addition $\mu$ is $\sigma$-finite, then the map is onto the subspace of $\mathcal{M}(X, \Sigma)$ consisting of complex measures which are absolutely continous with respect to $\mu$.*

*Proof.* The first assertion follows from Corollary 7.3 and Theorem 6.3. The second is a recapitulation of Theorem 5.1. □

## References

[Monygham, 2018] Monygham (2018). Introduction to measure theory. *github repository: "Monygham/Pedo-mellon-a-minno"*.

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