Analiza matematyczna w zadaniach, t. 1, W. Krysicki, L. Włodarski - rozwiązania

Całki nieoznaczone. Całkowanie przez podstawienie i całkowanie przez części.

15.22 $\int \left(5x^2 - 6x + 3 - \frac{2}{x} + \frac{5}{x^2}\right) dx$

15.23 $\int \frac{(x^2-1)^3}{x} dx$

15.24 $\int (x^2 - x + 1)(x^2 + x + 1) dx$

15.25 $\int (x^2+4)^5 x dx$

15.26 $\int \frac{x dx}{1+x^2}$

15.27 $\int \frac{x dx}{(x^2+3)^6}$

15.28 $\int \frac{x^2 dx}{a^3+x^3}$, $a \neq 0$, $x \neq -a$

15.29 $\int \frac{x\sqrt[3]{x} + \sqrt[4]{x}}{x^2} dx$

15.30 $\int \frac{x\sqrt{x} - x\sqrt[4]{x}}{\sqrt[3]{x}} dx$

15.31 $\int (3 + 2\sqrt[4]{x})^3 dx$

15.32 $\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 4\sqrt[4]{5x^3}}{6\sqrt[3]{x}} dx$

15.33 $\int \frac{3 + 5\sqrt[3]{x^2}}{\sqrt{x^3}} dx$

15.34 $\int \sqrt{3x+1}\, dx$

15.35 $\int \sqrt{a+bx} dx$

15.36 $\int \frac{x dx}{\sqrt[3]{2x^2-1}}$

15.37 $\int x\sqrt{1+x^2} dx$

15.38 $\int \frac{x}{\sqrt{3-5x^2}} dx$

15.39 $\int \frac{x-1}{\sqrt[3]{x+1}} dx$

15.40 $\int \frac{x}{\sqrt{x^2-6}} dx$

15.41 $\int \frac{x^2}{\sqrt[5]{x^3+1}} dx$

15.42 $\int \frac{e^{\frac{1}{x}}}{x^2} dx$

15.43 $\int x e^{-x^2} dx$

15.44 $\int \frac{dx}{2\cos^2 3x}$

15.45 $\int x \sin(2x^2+1) dx$

15.46 $\int \sin^5 x \cos x\, dx$

15.47 $\int \frac{\cos x}{\sqrt{1+\sin x}} dx$

15.48 $\int \frac{\sin x}{a + b\cos x} dx$, $b \neq 0$

15.49 $\int \cos x \cdot e^{\sin x} dx$

15.50 $\int \frac{x^3 dx}{\cos^2 x^4}$

15.51 $\int \frac{\tan x}{\cos^2 x} dx$

15.52 $\int \frac{x^2 dx}{\cos^2(x^3+1)}$

15.53 $\int \frac{(\ln x)^2}{x} dx$

15.54 $\int \frac{dx}{e^x + e^{-x}}$

15.55 $\int \frac{e^x dx}{2e^x + 1}$

15.56 $\int x \ln(1+x^2) dx$

15.57 $\int \frac{\sqrt{2 + \ln|x|}}{x} dx$

15.58 $\int 6^{1-x} dx$

15.59 $\int \frac{dx}{x\sqrt{1-\ln^2|x|}}$

15.60 $\int \frac{\ln|\arctan x| dx}{1+x^2}$

15.61 $\int x e^{x^2}(x^2+1) dx$

15.62 $\int \frac{x^2 dx}{\sqrt{1-x^6}}$

15.63 $\int \frac{dx}{(1+x^2)\arctan x}$

15.64 $\int \frac{(\pi - \arcsin x) dx}{\sqrt{1-x^2}}$

15.65 $\int \frac{x dx}{x^4+1}$

15.66 $\int x^4(1+x)^3 dx$

15.67 $\int x^2 e^x dx$

15.68 $\int x^3 e^x dx$

15.69 $\int x^4 e^{2x} dx$

15.70 $\int x\cos x dx$

15.71 $\int x^2\cos x dx$

15.72 $\int x^2\sin 5x dx$

15.73 $\int e^x\cos x dx$

15.74 $\int e^{-2x}\sin 3x dx$

15.75 $\int e^x\cos(\frac{2}{3}x) dx$

15.76 $\int \sqrt{x}\ln x dx$

15.77 $\int (\ln|x|)^3 dx$

15.78 $\int \frac{(\ln|x|)^2}{x^5} dx$

15.79 $\int \sqrt{x}(\ln|x|)^3 dx$

15.80 $\int \frac{\ln|x|}{x^4} dx$

15.81 $\int \frac{(\ln x)^2}{\sqrt{x}} dx$

15.82 $\int x^3(\ln x)^2 dx$

15.83 $\int x^n\ln x\ dx, n\neq -1$

Całki funkcji wymiernych.

16.26 $\int (2x+1)^3 dx$

16.27 $\int \frac{dx}{(3x-2)^4}$

16.28 $\int \frac{3x-4}{x^2-x-6} dx$

16.29 $\int \frac{2x-3}{x^2-3x+3} dx$

16.30 $\int \frac{x+13}{x^2-4x-5} dx$

16.31 $\int \frac{2x+6}{2x^2+3x+1} dx$

16.32 $\int \frac{6x-13}{x^2-\frac{7}{2}x+\frac{3}{2}} dx$

16.33 $\int \frac{4x-5}{2x^2-5x+3} dx$

16.34 $\int \frac{5x+11}{x^2+3x-10} dx$

16.35 $\int \frac{\frac{5}{6}x-16}{x^2+3x-18} dx$

16.36 $\int \frac{dx}{x^2+2x-1}$

16.37 $\int \frac{dx}{6x^2-13x+6}$

16.38 $\int \frac{5+x}{10x+x^2} dx$

16.39 $\int \frac{7x}{4+5x^2} dx$

16.40 $\int \frac{dx}{-5+6x-x^2}$

16.41 $\int \frac{dx}{1+x-x^2}$

16.42 $\int \frac{dx}{2x-3x^2}$

16.43 $\int \frac{3x+2}{x^2-x-2} dx$

16.44 $\int \frac{2x-1}{x^2-6x+9} dx$

16.45 $\int \frac{x-1}{4x^2-4x+1} dx$

16.46 $\int \frac{2x-13}{(x-5)^2} dx$

16.47 $\int \frac{3x+1}{(x+2)^2} dx$

16.48 $\int \frac{dx}{2x^2-2x+5}$

16.49 $\int \frac{dx}{3x^2+2x+1}$

16.50 $\int \frac{dx}{13-6x+x^2}$

16.51 $\int \frac{3dx}{9x^2-6x+2}$

16.52 $\int \frac{x+1}{x^2-x+1} dx$

16.53 $\int \frac{4x-1}{2x^2-2x+1} dx$

16.54 $\int \frac{2x-1}{x^2-2x+5} dx$

16.55 $\int \frac{2x-10}{x^2-2x+10} dx$

16.56 $\int \frac{2x-20}{x^2-8x+25} dx$

16.57 $\int \frac{3x+4}{x^2+4x+8} dx$

16.58 $\int \frac{x+6}{x^2-3} dx$

16.59 $\int \frac{x+6}{x^2+3} dx$

16.60 $\int \frac{6x}{x^2+4x+13} dx$

16.61 $\int \frac{10x-44}{x^2-4x+20} dx$

16.62 $\int \frac{4x-5}{x^2-6x+10} dx$

16.63 $\int \frac{5x}{2+3x} dx$

16.64 $\int \frac{x^2}{5x^2+12}dx$

16.65 $\int \frac{2x^2+7x+20}{x^2+6x+25}dx$

16.66 $\int \frac{7x^2+7x-176}{x^3-9x^2+6x+56}dx$

16.67 $\int \frac{x^3-4x^2+1}{(x-2)^4}dx$

16.68 $\int \frac{3x^2-5x+2}{x^3-2x^2+3x-6}dx$

16.69 $\int \frac{2x+1}{(x^2+1)^2}dx$

16.70 $\int \frac{x^3+2x-6}{x^2-x-2}dx$

16.71 $\int \frac{2x^3-19x^2+58x-42}{x^2-8x+16}dx$

16.72 $\int \frac{x^4}{x^2+1}dx$

16.73 $\int \frac{72x^6}{3x^2+2}dx$

16.74 $\int \frac{2x^4-10x^3+21x^2-20x+5}{x^2-3x+2}dx$

16.75 $\int \frac{x^2+5x+41}{(x+3)(x-1)(x-\frac{1}{2})}dx$

16.76 $\int \frac{17x^2-x-26}{(x^2-1)(x^2-4)}dx$

16.77 $\int \frac{2x}{(x^2+1)(x^2+3)}dx$

16.78 $\int \frac{10x^3+110x+400}{(x^2-4x+29)(x^2-2x+5)}dx$

16.79 $\int \frac{4x^3-2x^2+6x-13}{x^4+3x^2-4}dx$

16.80 $\int \frac{10x^3+40x^2+40x+6}{x^4+6x^3+11x^2+6x}dx$

16.81 $\int \frac{6x^3+4x+1}{x^4+x^2}dx$

16.82 $\int \frac{dx}{x^3-a^2x}$

16.83 $\int \frac{dx}{x^3+x^2+x}$

16.84 $\int \frac{dx}{x^4+x^2+1}$

16.85 $\int \frac{5x^3+3x^2+12x-12}{x^4-16}dx$

16.86 $\int \frac{15x^2+66x+21}{(x-1)(x^2+4x+29)}dx$

16.87 $\int \frac{4x^3+9x^2+4x+1}{x^4+3x^3+3x^2+x}dx$

16.88 $\int \frac{dx}{x^3(x-1)^2(x+1)}$

16.89 $\int \frac{dx}{(x^2+x+1)^2}$

16.90 $\int \frac{3x^2-17x+21}{(x-2)^3}dx$

16.91 $\int \frac{dx}{(x^2+4x+8)^3}$

16.92 $\int \frac{x^3-2x^2+7x+4}{(x-1)^2(x+1)^2}dx$

16.93 $\int \frac{dx}{x^4+64}$

16.94 $\int \frac{5x^3-11x^2+5x+4}{(x-1)^4}dx$

16.95 $\int \frac{dx}{x^4+6x^2+25}$

16.96 $\int \frac{9x^4-3x^3-23x^2+30x-1}{(x-1)^4(x+3)}dx$

16.97 $\int \frac{x^3-2x^2+5x-8}{x^4+8x^2+16}dx$

16.98 $\int \frac{3x^2+x-2}{(x-1)^3(x^2+1)}dx$

Całki funkcji niewymiernych. Całki funkcji zawierających pierwiastki z wyrażenia liniowego.

17.6 $\int \sqrt{2x+1}dx$

17.7 $\int \frac{dx}{\sqrt{3+4x}}$

17.8 $\int \frac{dx}{\sqrt[3]{3x-4}}$

17.9 $\int \frac{dx}{\sqrt[5]{(2x+1)^3}}$

17.10 $\int x\sqrt[3]{x-4}dx$

17.11 $\int x\sqrt[3]{3x-1}dx$

17.12 $\int x\sqrt{2+3x}dx$

17.13 $\int x\sqrt{1-5x}dx$

17.14 $\int x\sqrt[3]{x-4}dx$

17.15 $\int \frac{xdx}{\sqrt[4]{2x+3}}$

17.16 $\int \frac{x^2dx}{3\sqrt[3]{x+2}}$

17.17 $\int \frac{x^2+1}{\sqrt{3x+1}}dx$

17.18 $\int x\sqrt[4]{2x+3}dx$

17.19 $\int \frac{dx}{x\sqrt{x+a}}$

17.20 $\int \frac{dx}{x\sqrt{x-a}}$

17.21 $\int \frac{\sqrt{x}}{x-1}dx$

17.22 $\int \frac{\sqrt{x+1}}{x}dx$

17.23 $\int \frac{1+\sqrt{x}}{1-\sqrt{x}}dx$

17.24 $\int \frac{dx}{(x+1)\sqrt{1-x}}$

17.25 $\int \sqrt{1+\sqrt{x}}dx$

17.26 $\int \frac{\sqrt[3]{x}dx}{x+\sqrt[6]{x^5}}$

17.27 $\int \frac{dx}{\sqrt{x}+2\sqrt[3]{x^2}}$

17.28 $\int \frac{dx}{\sqrt{x-5}+\sqrt{x-7}}$

17.29 $\int \frac{dx}{x\sqrt{x+9}}$

17.30 $\int x^2\sqrt[3]{7-2x}dx$

17.31 $\int \frac{dx}{\sqrt{x+1}+\sqrt[3]{x+1}}$

17.32 $\int \sqrt{\frac{x-1}{x-2}}\cdot\frac{dx}{(x-1)^2}$

17.33 $\int \sqrt{\frac{1-x}{1+x}}\cdot\frac{dx}{x}$

17.34 $\int \frac{xdx}{\sqrt[3]{x+1}-\sqrt{x+1}}$

17.35 $\int \frac{\sqrt[3]{x^2}-\sqrt{x}+1}{\sqrt[3]{x}-1}dx$

Całki funkcji zawierających pierwiastek kwadratowy z trójmianu kwadratowego

17.51 $\int \frac{(8x+3)dx}{\sqrt{4x^2+3x+1}}$

17.52 $\int \frac{(10x+15)dx}{\sqrt{36x^2+108x+77}}$

17.53 $\int \frac{dx}{\sqrt{2x-x^2}}$

17.54 $\int \frac{dx}{\sqrt{7-6x-x^2}}$

17.55 $\int \frac{dx}{\sqrt{1-9x^2}}$

17.56 $\int \frac{dx}{\sqrt{(2r-x)x}}$

17.57 $\int \frac{(x+3)dx}{\sqrt{1-4x^2}}$

17.58 $\int \frac{xdx}{\sqrt{1-2x-3x^2}}$

17.59 $\int \sqrt{1-4x^2}dx$

17.60 $\int \frac{6x+5}{\sqrt{6+x-x^2}}dx$

17.61 $\int \frac{x-5}{\sqrt{5+4x-x^2}}dx$

17.62 $\int \frac{x+1}{\sqrt{8+2x-x^2}}dx$

17.63 $\int \sqrt{6x-x^2}dx$

17.64 $\int \frac{2x-3}{\sqrt{3-2x-x^2}}dx$

17.65 $\int \frac{dx}{\sqrt{x^2+3x+2}}$

17.66 $\int \frac{dx}{\sqrt{4x^2+3x-1}}$

17.67 $\int \frac{dx}{\sqrt{x^2-x+m}}$

17.68 $\int \frac{dx}{\sqrt{(x-a)(x-3a)}}$

17.69 $\int \frac{(x+3)dx}{\sqrt{x^2+2x}}$

17.70 $\int \frac{(3x+2)dx}{\sqrt{x^2-5x+19}}$

17.71 $\int \frac{x+a}{\sqrt{x^2-ax}}dx$

17.72 $\int \frac{3x-2}{\sqrt{4x^2-4x+5}}dx$

17.73 $\int \frac{3x+2}{\sqrt{x^2-4x+5}}dx$

17.74 $\int \frac{3x-4}{\sqrt{4x^2+5x-8}}dx$

17.75 $\int \frac{5x+2}{\sqrt{2x^2+8x-1}}dx$

17.76 $\int \sqrt{2x+x^2}dx$

17.77 $\int \frac{5x-4}{\sqrt{3x^2-2x+1}}dx$

17.78 $\int \sqrt{3-2x-x^2}dx$

17.79 $\int \sqrt{x^2-4}dx$

17.80 $\int \sqrt{3x^2+10x+9}dx$

17.81 $\int \sqrt{x^2-3x+2}dx$

17.82 $\int \frac{x^2dx}{\sqrt{1-x^2}}$

17.83 $\int \frac{x^2dx}{\sqrt{x^2+2x+2}}$

17.84 $\int \sqrt{\frac{x}{1-x}}dx$

17.85 $\int \frac{2ax^2+1}{\sqrt{ax^2+2x+1}}dx$

17.86 $\int \frac{2x^2+3x+1}{\sqrt{x^2+1}}dx$

17.87 $\int \frac{2x^2-ax+a^2}{\sqrt{x^2+a^2}}dx$

17.88 $\int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx$

17.89 $\int \frac{x^3+2x^2+x-1}{\sqrt{x^2+2x-1}}dx$

17.90 $\int \frac{x^3dx}{\sqrt{x^2-4x+3}}$

17.91 $\int \frac{3x^3+2}{\sqrt{x^2+x+1}}dx$

17.92 $\int x^2\sqrt{4x-x^2}dx$

17.93 $\int x\sqrt{6+x-x^2}dx$

17.94 $\int \frac{x^4dx}{\sqrt{5x^2+4}}$

17.95 $\int \frac{x^3+5x^2-3x+4}{\sqrt{x^2+x+1}}dx$

17.96 $\int \frac{5x^2-2x+10}{\sqrt{3x^2-5x+8}}dx$

17.97 $\int \frac{x^3+4x^2-6x+3}{\sqrt{5+6x-x^2}}dx$

17.98 $\int x\sqrt{8+x-x^2}dx$

17.99 $\int (2x-5)\sqrt{2+3x-x^2}dx$

17.100 $\int \frac{x^3dx}{\sqrt{2x^2+3}}$

17.101 $\int \frac{x^5dx}{\sqrt{2x^2+3}}$

17.102 $\int \frac{x^4dx}{\sqrt{3+2x+x^2}}$

17.103 $\int \frac{dx}{x\sqrt{10x-x^2}}$

17.104 $\int \frac{\mathrm{d}x}{(x+1)\sqrt{x^2-1}}$

17.105 $\int \frac{\mathrm{d}x}{(x+2)\sqrt{4-x^2}}$

17.106 $\int \frac{\mathrm{d}x}{x\sqrt{x^2+x-1}}$

17.107 $\int \frac{\mathrm{d}x}{x\sqrt{x^2-2x-1}}$

17.108 $\int \frac{\mathrm{d}x}{(2x-1)\sqrt{x^2-1}}$

17.109 $\int \frac{\mathrm{d}x}{(x+1)\sqrt{1+2x-3x^2}}$

17.110 $\int \frac{\mathrm{d}x}{(3-2x)\sqrt{x^2-4x+3}}$

17.111 $\int \frac{\mathrm{d}x}{x\sqrt{x^2+x+1}}$

17.112 $\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$

17.113 $\int \frac{\mathrm{d}x}{(a-x)\sqrt{a^2-x^2}}$

17.114 $\int \frac{\mathrm{d}x}{(x-2)\sqrt{x^2-6x+1}}$

17.115 $\int \frac{\mathrm{d}x}{x^2\sqrt{4-x^2}}$

17.116 $\int \frac{\mathrm{d}x}{(x-1)^2\sqrt{10x-x^2}}$

17.117 $\int \frac{\mathrm{d}x}{x^3\sqrt{x^2+1}}$

17.118 $\int \frac{\mathrm{d}x}{x^3\sqrt{2x^2+2x+1}}$

17.119 $\int \frac{\mathrm{d}x}{(x-1)^3\sqrt{3-2x^2}}$

17.120 $\int \frac{\mathrm{d}x}{x^2\sqrt{1-4x+x^2}}$

17.121 $\int \frac{\mathrm{d}x}{x^3\sqrt{1+x^2}}$

17.122 $\int \frac{\mathrm{d}x}{x^4\sqrt{3-2x+x^2}}$

17.123 $\int \frac{\mathrm{d}x}{(x-2)^4\sqrt{1-4x+x^2}}$

Całki funkcji trygonometrycznych.

18.30 $\int \cos 5x \cos 7x dx$

18.31 $\int \sin 3x \cos 2x dx$

18.32 $\int \cos 2x \cos 3x dx$

18.33 $\int \sin x \cos 3x dx$

18.34 $\int \cos 2x \sin 4x dx$

18.35 $\int \sin 2x \sin 5x dx$

18.36 $\int \cos x \cos 3x dx$

18.37 $\int \sin 3x \sin x dx$

18.38 $\int \sin 5x \sin 2x dx$

18.39 $\int \sin^3 x dx$

18.40 $\int \sin^4 x dx$

18.41 $\int \cos^4 x dx$

18.42 $\int \cos^5 x dx$

18.43 $\int \sin^5 x dx$

18.44 $\int \tan^5 x dx$

18.45 $\int \cot^4 x dx$

18.46 $\int ctg^6 x dx$

18.47 $\int \sin^3 x \cos^4 x dx$

18.48 $\int \sin^7 x \cos^6 x dx$

18.49 $\int \sin^5 x \cos^2 x dx$

18.50 $\int \sin^2 x \cos^2 x dx$

18.51 $\int \sin^3 x \cos^3 x dx$

18.52 $\int \sin^4 x \cos^5 x dx$

18.53 $\int \frac{\cos x dx}{\sin^8 x}$

18.54 $\int \sin x \tan x dx$

18.55 $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx$

18.56 $\int \frac{\sin x dx}{\sqrt[3]{1 + 2\cos x}}$

18.57 $\int \frac{\sin 2x dx}{\sqrt{1 + \cos^2 x}}$

18.58 $\int \frac{\sin 2x}{1 + \sin^2 x} dx$

18.59 $\int \frac{\sin 2x dx}{\sqrt{1 - sin^4 x}}$

18.60 $\int \frac{\cos^3 x}{\sin^2 x} dx$

18.61 $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x - \sin x \cos x + \cos^2 x} dx$

18.62 $\int \frac{dx}{\sin^3 x}$

18.63 $\int \frac{dx}{\cos^3 x}$

18.64 $\int \frac{dx}{\sin^4 x}$

18.65 $\int \frac{dx}{\cos^5 x}$

18.66 $\int \frac{dx}{\sin^7 x}$

18.67 $\int \frac{dx}{\sin^3 x \cos x}$

18.68 $\int \frac{dx}{\sin x \cos^3 x}$

18.69 $\int \frac{dx}{\sin^5 x \cos^3 x}$

18.70 $\int \frac{dx}{\sin^2 x \cos^4 x}$

18.71 $\int \frac{\sin^4 x}{\cos^3 x} dx$

18.72 $\int \frac{\sin^4 x dx}{\cos x}$

18.73 $\int \frac{\cos^5 x dx}{\sin^3 x}$

18.74 $\int \frac{\sin^3 x dx}{\cos^8 x}$

18.75 $\int \frac{\cos 2x dx}{\cos^3 x}$

18.76 $\int \frac{dx}{5 + 4\cos x}$

18.77 $\int \frac{dx}{1 + \sin x}$

18.78 $\int \frac{dx}{\sin x + \cos x}$

18.79 $\int \frac{\sin x \cos x dx}{\sin^4 x + \cos^4 x}$

18.80 $\int \frac{3 + \sin^2 x}{2\cos^2 x - \cos^4 x} dx$

18.81 $\int \frac{\cos x + \sin x}{(\sin x - \cos x)^2} dx$

18.82 $\int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx$

18.83 $\int \frac{\sin x \cos x}{1 + \sin^4 x} dx$

18.84 $\int \frac{dx}{(\sin^2 x + 3\cos^2 x)^2}$

18.85 $\int \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x} dx$

18.86 $\int \frac{dx}{\sin^4 x + \cos^4 x}$

18.87 $\int \frac{dx}{1 - \sin^4 x}$

Całki funkcji cyklometrycznych.

18.91 $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}} \arcsin x dx$

18.92 $\displaystyle\int \frac{\arcsin x}{\sqrt{(1-x^2)^3}} dx$

18.93 $\displaystyle\int \frac{x^2}{1+x^2} \arctan x dx$

18.94 $\displaystyle\int \frac{dx}{(1+9x^2)\sqrt{\arctan 3x}}$

18.95 $\displaystyle\int \frac{dx}{(1+4x^2)(\arctan 2x)^2}$

18.96 $\displaystyle\int \frac{(\arctan x)^2}{x^2+1} dx$

18.97 $\displaystyle\int \frac{dx}{\sqrt{1-x^2} \arccos^2 x}$

18.98 $\displaystyle\int \frac{dx}{\sqrt{1-x^2} \arcsin x}$

18.99 $\displaystyle\int \frac{x \arctan x dx}{(1+x^2)^2}$

18.100 $\displaystyle\int \frac{x \arcsin x dx}{(1-x^2)^{\frac{3}{2}}}$

18.101 $\displaystyle\int x \arcsin x dx$

18.102 $\displaystyle\int \frac{x \arctan x dx}{(x^2-1)^2}$

18.103 $\displaystyle\int x^2 \arctan x dx$

18.104 $\displaystyle\int \frac{\arctan e^{\frac{1}{2}x}}{e^{\frac{1}{2}x}(1+e^x)} dx$

18.105 $\displaystyle\int \frac{\arcsin x dx}{x^2}$

18.106 $\displaystyle\int \frac{\arcsin e^x}{e^x} dx$

18.107 $\displaystyle\int x^3 \arctan x dx$

18.108 $\displaystyle\int (2x+3) \arccos(2x-3) dx$

18.109 $\displaystyle\int \frac{x \arctan x}{\sqrt{1+x^2}} dx$

18.110 $\displaystyle\int \sqrt{1-x^2} \arcsin x dx$

18.111 $\displaystyle\int x(1+x^2) \arctan x dx$

18.112 $\displaystyle\int \arcsin \frac{2\sqrt{x}}{1+x} dx$

Całki funkcji wykładniczych i logarytmicznych.

18.118 $\displaystyle\int (e^{3x} + \sqrt{e^x}) dx$

18.119 $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

18.120 $\displaystyle\int \frac{dx}{e^{2x}-1}$

18.121 $\displaystyle\int \frac{dx}{e^x + e^{-x}}$

18.122 $\displaystyle\int \sqrt{e^x+1} dx$

18.123 $\displaystyle\int \frac{e^x-1}{e^x+1} dx$

18.124 $\displaystyle\int \frac{dx}{\sqrt{3+2e^x}}$

18.125 $\displaystyle\int e^x \sqrt{1+e^x} dx$

18.126 $\displaystyle\int \frac{e^x}{(e^x-1)^2} dx$

18.127 $\displaystyle\int (e^x + e^{-x})^2 dx$

18.128 $\displaystyle\int \frac{e^x}{e^x+5} dx$

18.129 $\displaystyle\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx$

18.130 $\displaystyle\int \frac{dx}{e^x + e^{2x}}$

18.131 $\displaystyle\int \frac{e^x}{(e^x+a)^n} dx$

18.132 $\displaystyle\int \frac{e^x dx}{\sqrt{3-5e^{2x}}}$

18.133 $\displaystyle\int \frac{dx}{\sqrt{e^{2x}+4e^x+1}}$

18.134 $\displaystyle\int x^3 e^{-x} dx$

18.135 $\displaystyle\int \frac{dx}{x \ln x}$

18.136 $\displaystyle\int \ln(x^2+1) dx$

18.137 $\displaystyle\int (\ln|x|)^2 dx$

18.138 $\displaystyle\int \ln(x + \sqrt{x^2+1}) dx$

18.139 $\displaystyle\int \ln|2+5x| dx$

18.140 $\displaystyle\int \frac{dx}{x(1+\ln^2|x|)}$

18.141 $\displaystyle\int x^{-2} \ln|x| dx$

18.142 $\displaystyle\int (4+3x)^2 \ln|x| dx$

18.143 $\displaystyle\int x^3 \ln(x^2+3) dx$

18.144 $\displaystyle\int x a^x dx,\ a>1$

15 Całki nieoznaczone. Całkowanie przez podstawienie i całkowanie przez części.

15.22

$$\int \left(5x^2 - 6x + 3 - \frac{2}{x} + \frac{5}{x^2}\right) = \frac{5}{3}x^3 - 3x^2 + 3x - 2\ln|x| - \frac{5}{x} + C$$

Podstawowe prawa całkowania i wzory na całki funkcji elementarnych:
Całka z iloczynu funkcji przez stałą:

$$\int a \cdot f(x)dx = a \cdot \int f(x)dx, \text{ gdzie } a \in R$$

Całka z sumy (różnicy) funkcji:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\int ax^n dx = \frac{a}{n+1}x^{n+1} + C, \text{ gdzie } a \in R \wedge n \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + C$$

15.23

$$\int \frac{(x^2-1)^3}{x}dx = \int \frac{x^6 - 3x^4 + 3x^2 - 1}{x}dx = \int \left(x^5 - 3x^3 + 3x - \frac{1}{x}\right)dx =$$
$$= \frac{1}{6}x^6 - \frac{3}{4}x^4 + \frac{3}{2}x^2 - \ln|x| + C$$

15.24

$$\int (x^2 - x + 1)(x^2 + x + 1)dx = \int (x^4 + x^2 + 1)dx = \frac{1}{5}x^5 + \frac{1}{3}x^3 + x + C$$

15.25

$$\int (x^2+4)^5 x dx = \begin{vmatrix} t = x^2 + 4 \\ dt = 2xdx \\ \frac{1}{2}dt = xdx \end{vmatrix} = \frac{1}{2}\int t^5 dt = \frac{1}{12}t^6 + C = \frac{1}{12}(x^2+4)^6 + C$$

15.26

$$\int \frac{xdx}{1+x^2} = \begin{vmatrix} t = 1 + x^2 \\ dt = 2xdx \\ \frac{1}{2}dt = xdx \end{vmatrix} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|1+x^2| + C$$

15.27

$$\int \frac{xdx}{(x^2+3)^6} = \begin{vmatrix} t = x^2+3 \\ dt = 2xdx \\ \frac{1}{2}dt = xdx \end{vmatrix} = \frac{1}{2}\int \frac{dt}{t^6} = \frac{1}{2}\int t^{-6}dt = -\frac{1}{10}t^{-5} + C = \frac{-1}{10(x^2+3)^5} + C$$

15.28

$$\int \frac{x^2dx}{a^3+x^3},\ a \neq 0,\ x \neq -a$$

$$= \begin{vmatrix} t = a^3+x^3 \\ dt = 3x^2dx \\ \frac{1}{3}dt = x^2dx \end{vmatrix} = \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3}\ln|t| + C = \frac{1}{3}\ln|a^3+x^3| + C$$

15.29

$$\int \frac{x\sqrt[3]{x} + \sqrt[4]{x}}{x^2}dx = \int \frac{x^{\frac{4}{3}} + x^{\frac{1}{4}}}{x^2}dx = \int \left(x^{-\frac{2}{3}} + x^{-\frac{7}{4}}\right)dx = 3x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{3}{4}} + C$$

15.30

$$\int \frac{x\sqrt{x} - x\sqrt[4]{x}}{\sqrt[3]{x}}dx = \int \frac{x^{\frac{3}{2}} - x^{\frac{5}{4}}}{x^{\frac{1}{3}}}dx = \int \left(x^{\frac{7}{6}} - x^{\frac{11}{12}}\right)dx = \frac{6}{13}x^{\frac{13}{6}} - \frac{12}{23}x^{\frac{23}{12}} + C$$

15.31

$$\int (3+2\sqrt[4]{x})^3dx = \int \left(27 + 54x^{\frac{1}{4}} + 36x^{\frac{1}{2}} + 8x^{\frac{3}{4}}\right)dx = 27x + \frac{216}{5}x^{\frac{5}{4}} + 24x^{\frac{3}{2}} + \frac{32}{7}x^{\frac{7}{4}} + C$$

15.32

$$\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 4\sqrt[4]{5x^3}}{6\sqrt[3]{x}}dx = \int \frac{x^{\frac{1}{2}} - 2x^{\frac{2}{3}} + 4\sqrt[4]{5}\cdot x^{\frac{3}{4}}}{6x^{\frac{1}{3}}}dx =$$

$$= \frac{1}{6}\int x^{\frac{1}{6}}dx - \frac{1}{3}\int x^{\frac{1}{3}}dx + \frac{2\sqrt[3]{5}}{3}\int x^{\frac{5}{12}}dx = \frac{1}{7}x^{\frac{7}{6}} - \frac{1}{4}x^{\frac{4}{3}} + \frac{8\sqrt[4]{5}}{17}x^{\frac{17}{12}} + C$$

15.33

$$\int \frac{3+5\sqrt[3]{x^2}}{\sqrt{x^3}}dx = 3\int x^{-\frac{3}{2}}dx + 5\int x^{-\frac{5}{6}}dx = -6x^{-\frac{1}{2}} + 30x^{\frac{1}{6}} + C = \frac{-6}{\sqrt{x}} + 30\sqrt[6]{x} + C$$

15.34

$$\int \sqrt{3x+1}\mathrm{d}x = \begin{vmatrix} t = 3x+1 \\ dt = 3dx \\ \frac{1}{3}dt = dx \end{vmatrix} = \frac{1}{3}\int t^{\frac{1}{2}}dt = \frac{2}{9}t^{\frac{3}{2}} + C = \frac{2}{9}(3x+1)^{\frac{3}{2}} + C$$

15.35

$$\int \sqrt{a+bx}dx = \begin{vmatrix} t = a+bx \\ dt = bdx \\ \frac{1}{b}dt = dx \end{vmatrix} = \frac{1}{b}\int t^{\frac{1}{2}}dt = \frac{2}{3b}t^{\frac{3}{2}} + C = \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C, \text{ gdzie } b \neq 0$$

15.36

$$\int \frac{xdx}{\sqrt[3]{2x^2-1}} = \begin{vmatrix} t = 2x^2-1 \\ dt = 4xdx \\ \frac{1}{4}dt = xdx \end{vmatrix} = \frac{1}{4}\int t^{-\frac{1}{3}}dt = \frac{3}{8}t^{\frac{2}{3}} + C = \frac{3}{8}(2x^2-1)^{\frac{2}{3}} + C, \text{ gdzie } x \neq \frac{1}{\sqrt{2}}$$

15.37

$$\int x\sqrt{1+x^2}dx = \begin{vmatrix} t = 1+x^2 \\ dt = 2xdx \\ \frac{1}{2}dt = xdx \end{vmatrix} = \frac{1}{2}\int t^{\frac{1}{2}}dt = \frac{1}{3}t^{\frac{3}{2}} + C = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

15.38

$$\int \frac{x}{\sqrt{3-5x^2}}dx = \begin{vmatrix} t = 3-5x^2 \\ dt = -10xdx \\ -\frac{1}{10}dt = xdx \end{vmatrix} = -\frac{1}{10}\int t^{-\frac{1}{2}}dt = -\frac{1}{5}t^{\frac{1}{2}} + C = -\frac{1}{5}\sqrt{3-5x^2} + C$$

15.39

$$\int \frac{x-1}{\sqrt[3]{x+1}}dx = \int \frac{(x+1)-2}{\sqrt[3]{x+1}}dx = \int (x+1)^{\frac{2}{3}}dx - 2\int (x+1)^{-\frac{1}{3}}dx =$$
$$= \frac{3}{5}(x+1)^{\frac{5}{3}} - 3(x+1)^{\frac{2}{3}} + C = \frac{3}{5}(x+1)(x+1)^{\frac{2}{3}} - 3(x+1)^{\frac{2}{3}} + C =$$
$$= \frac{3}{5}(x+1-5)(x+1)^{\frac{2}{3}} = \frac{3}{5}(x-4)(x+1)^{\frac{2}{3}} + C$$

$$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C, \text{ dla } n \neq -1$$

15.40

$$\int \frac{x}{\sqrt{x^2-6}}dx = \int \frac{2x}{2\sqrt{x^2-6}}dx = \sqrt{x^2-6} + C$$

$$\int \frac{f'(x)}{2\sqrt{f(x)}}dx = \sqrt{f(x)} + C$$

15.41

$$\int \frac{x^2}{\sqrt[5]{x^3+1}}dx = \begin{vmatrix} t = x^3+1 \\ dt = 3x^2dx \\ \frac{1}{3}dt = x^2dx \end{vmatrix} = \frac{1}{3}\int t^{-\frac{1}{5}}dt = \frac{5}{12}t^{\frac{4}{5}} + C = \frac{5}{12}(x^3+1)^{\frac{4}{5}} + C$$

15.42

$$\int \frac{e^{\frac{1}{x}}}{x^2}dx = \begin{vmatrix} t = \frac{1}{x} \\ dt = \frac{-dx}{x^2} \\ -dt = \frac{dx}{x^2} \end{vmatrix} = -\int e^t dt = -e^t + C = -e^{\frac{1}{x}} + C$$

15.43

$$\int xe^{-x^2} = \begin{vmatrix} t = -x^2 \\ dt = -2xdx \\ -\frac{1}{2}dt = xdx \end{vmatrix} = -\frac{1}{2}\int e^t dt = -\frac{1}{2}e^t + C = -\frac{1}{2}e^{-x^2} + C$$

15.44

$$\int \frac{dx}{2\cos^2 3x} = \begin{vmatrix} t = 3x \\ dt = 3dx \\ \frac{1}{3}dt = dx \end{vmatrix} = \frac{1}{6}\int \frac{dt}{\cos^2 t} = \frac{1}{6}\tan t + C = \frac{1}{6}\tan 3x + C \int \frac{dx}{\cos^2 x} = \tan x + C$$

15.45

$$\int x\sin(2x^2+1)dx = \begin{vmatrix} t = 2x^2+1 \\ dt = 4xdx \\ \frac{1}{4}dt = xdx \end{vmatrix} = \frac{1}{4}\int \sin t dt = -\frac{1}{4}\cos t + C = -\frac{1}{4}\cos(2x^2+1) + C$$

15.46

$$\int \sin^5 x\cos x dx = \begin{vmatrix} t = \sin x \\ dt = \cos x\ dx \end{vmatrix} = \int t^5 dt = \frac{1}{6}t^6 + C = \frac{1}{6}\sin^6 x + C$$

15.47

$$\int \frac{\cos x}{\sqrt{1+\sin x}}dx = \begin{vmatrix} t = 1+\sin x \\ dt = \cos x\ dx \end{vmatrix} = \int t^{-\frac{1}{2}}dt = 2t^{\frac{1}{2}} + C = 2\sqrt{1+\sin x} + C$$

15.48

$$\int \frac{\sin x}{a + b\cos x}dx, b \neq 0$$

$$= \begin{vmatrix} t = a + b\cos x \\ dt = -b\sin x \ dx \\ -\frac{1}{b}dt = \sin x \ dx \end{vmatrix} = -\frac{1}{b}\int \frac{dt}{t} = -\frac{1}{b}\ln|t| + C = -\frac{1}{b}\ln|a + b\cos x| + C$$

15.49

$$\int \cos x \cdot e^{\sin x} = e^{\sin x} + C$$

$$\boxed{\int f'(x) \cdot e^{f(x)} = e^{f(x)} + C}$$

15.50

$$\int \frac{x^3 dx}{\cos^2 x^4} = \begin{vmatrix} t = x^4 \\ dt = 4x^3 dx \\ \frac{1}{4}dt = x^3 dx \end{vmatrix} = \frac{1}{4}\int \frac{dt}{\cos^2 t} = \frac{1}{4}\tan t + C = \frac{1}{4}\tan x^4 + C$$

15.51

$$\int \frac{\tan x}{\cos^2 x}dx = \begin{vmatrix} t = \tan x \\ dt = \frac{dx}{\cos^2 x} \end{vmatrix} = \int t dt = \frac{1}{2}t^2 + C = \frac{1}{2}\tan^2 x + C$$

15.52

$$\int \frac{x^2 dx}{\cos^2(x^3+1)} = \begin{vmatrix} t = x^3 + 1 \\ dt = 3x^2 dx \\ \frac{1}{3}dt = x^2 dx \end{vmatrix} = \frac{1}{3}\int \frac{dt}{\cos^2 t} = \frac{1}{3}\tan t + C = \frac{1}{3}\tan(x^3+1) + C$$

15.53

$$\int \frac{(\ln x)^2}{x}dx = \begin{vmatrix} t = \ln x \\ dt = \frac{dx}{x} \end{vmatrix} = \int t^2 dt = \frac{1}{3}t^3 + C = \frac{1}{3}(\ln x)^3 + C$$

15.54

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \begin{vmatrix} t = e^x \\ dt = e^x dx \end{vmatrix} = \int \frac{\mathrm{d}t}{t^2+1} = \arctan t + C = \arctan(e^x) + C$$

15.55

$$\int \frac{e^x dx}{2e^x + 1} = \begin{vmatrix} t = 2e^x + 1 \\ dt = 2e^x dx \\ \frac{1}{2}dt = e^x dx \end{vmatrix} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|2e^x + 1| + C$$

15.56

$$\int x\ln(1+x^2)dx = \begin{vmatrix} t=1+x^2 \\ dt=2xdx \\ \frac{1}{2}dt=xdx \end{vmatrix} = \frac{1}{2}\int \ln t\ dt = \begin{vmatrix} u=\ln t & dv=dt \\ du=\frac{dt}{t} & v=t \end{vmatrix} = \frac{1}{2}\left(t\ln t - \int dt\right) =$$
$$= \frac{1}{2}t\ln t - \frac{1}{2}t + C = \frac{1}{2}(1+x^2)\ln(1+x^2) - \frac{1}{2}(1+x^2) + C = \frac{1}{2}(1+x^2)\ln(1+x^2) - \frac{1}{2}x^2 + C$$

Uwaga: liczbę $-\frac{1}{2}$ włączono do stałej całkowania

Wzór całkowania przez części:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

15.57

$$\int \frac{\sqrt{2+\ln|x|}}{x}dx = \begin{vmatrix} t=2+\ln|x| \\ dt=\frac{dx}{x} \end{vmatrix} = \int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}} + C = \frac{2}{3}(2+\ln|x|)^{\frac{3}{2}} + C$$

15.58

$$\int 6^{1-x}dx = \begin{vmatrix} t=1-x \\ dt=-dx \\ -dt=dx \end{vmatrix} = -\int 6^t dt = -\frac{6^t}{\ln 6} + C = -\frac{6^{1-x}}{\ln 6} + C$$

15.59

$$\int \frac{dx}{x\sqrt{1-\ln^2|x|}} = \begin{vmatrix} t=\ln|x| \\ dt=\frac{dx}{x} \end{vmatrix} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin(\ln|x|) + C$$

15.60

$$\int \frac{\ln|\arctan x|dx}{1+x^2} = \begin{vmatrix} t=\arctan x \\ dt=\frac{dx}{1+x^2} \end{vmatrix} = \int \ln t\ dt = \begin{vmatrix} u=\ln t & dv=dt \\ du=\frac{dt}{t} & v=t \end{vmatrix} = t\ln t - \int dt =$$
$$= t\ln t - t + C = \arctan x[\ln(\arctan x) - 1] + C$$

$$\int \ln|x|\ dx = x(\ln|x| - 1) + C$$

15.61

$$\int xe^{x^2}(x^2+1)dx = \int \frac{xe^{x^2+1}(x^2+1)}{e}dx = \begin{vmatrix} t=x^2+1 \\ dt=2xdx \\ \frac{1}{2}dt=xdx \end{vmatrix} = \frac{1}{2e}\int te^t dt =$$

$$= \begin{vmatrix} u = t & dv = e^t dt \\ du = dt & v = e^t \end{vmatrix} = \frac{1}{2e}\left(te^t - \int e^t dt\right) + C = \frac{1}{2e}\left((x^2+1)e^{x^2+1} - e^{x^2+1}\right) + C =$$
$$= \frac{1}{2}x^2 e^{x^2} + C$$

15.62

$$\int \frac{x^2 dx}{\sqrt{1-x^6}} = \begin{vmatrix} t = x^3 \\ \frac{1}{3}dt = x^2 dx \end{vmatrix} = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\arcsin t + C = \frac{1}{3}\arcsin(x^3) + C$$

15.63

$$\int \frac{dx}{(1+x^2)\arctan x} = \begin{vmatrix} t = \arctan x \\ dt = \frac{dx}{1+x^2} \end{vmatrix} = \int \frac{dt}{t} = \ln|t| + C = \ln|\arctan x| + C$$

15.64

$$\int \frac{(\pi - \arcsin x)dx}{\sqrt{1-x^2}} = \begin{vmatrix} t = \arcsin x \\ dt = \frac{dx}{\sqrt{1-x^2}} \end{vmatrix} = \int (\pi - t)dt = \pi t - \frac{1}{2}t^2 + C =$$
$$= \pi \arcsin x - \frac{1}{2}\arcsin^2 x + C$$

15.65

$$\int \frac{xdx}{x^4+1} = \begin{vmatrix} t = x^2 \\ \frac{1}{2}dt = xdx \end{vmatrix} = \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\arctan t + C = \frac{1}{2}\arctan(x^2) + C$$

15.66

$$\int x^4(1+x)^3 dx = \int (x^4 + 3x^5 + 3x^6 + x^7)dx = \frac{1}{5}x^5 + \frac{1}{2}x^6 + \frac{3}{7}x^7 + \frac{1}{8}x^8 + C$$

15.67

$$\int x^2 e^x dx = \begin{vmatrix} u = x^2 & dv = e^x dx \\ du = 2xdx & v = e^x \end{vmatrix} = x^2 e^x - 2\int xe^x dx = \begin{vmatrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{vmatrix} =$$
$$= x^2 e^x - 2xe^x + 2\int e^x dx = x^2 e^x - 2xe^x + 2e^x + C = e^x(x^2 - 2x + 2) + C$$

15.68

$$\int x^3 e^x dx = \begin{vmatrix} u = x^3 & dv = e^x dx \\ du = 3x^2 dx & v = e^x \end{vmatrix} = x^3 e^x - 3\int x^2 e^x = \begin{vmatrix} u = x^2 & dv = e^x dx \\ du = 2xdx & v = e^x \end{vmatrix} =$$
$$= x^3 e^x - 3x^2 e^x + 6\int xe^x dx = \begin{vmatrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{vmatrix} = x^3 e^x - 3x^2 e^x + 6xe^x - 6\int e^x dx =$$
$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

15.69

$$\int x^4e^{2x}dx = \begin{vmatrix} u = x^4 & dv = e^{2x}dx \\ du = 4x^3dx & v = \frac{1}{2}e^{2x}\end{vmatrix} = \frac{1}{2}x^4e^{2x} - 2\int x^3e^{2x} = \begin{vmatrix} u = x^3 & dv = e^{2x}dx \\ du = 3x^2dx & v = \frac{1}{2}e^{2x}\end{vmatrix} =$$

$$= \frac{1}{2}x^4e^{2x} - x^3e^{2x} + 3\int x^2e^{2x} = \begin{vmatrix} u = x^2 & dv = e^{2x}dx \\ du = 2xdx & v = \frac{1}{2}e^{2x}\end{vmatrix} =$$

$$= \frac{1}{2}x^4e^{2x} - x^3e^{2x} + \frac{3}{2}x^2e^{2x} - 3\int xe^{2x}dx = \begin{vmatrix} u = x & dv = e^{2x}dx \\ du = dx & v = \frac{1}{2}e^{2x}\end{vmatrix} =$$

$$= \frac{1}{2}x^4e^{2x} - x^3e^{2x} + \frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{2}\int e^{2x}dx =$$

$$= \frac{1}{2}x^4e^{2x} - x^3e^{2x} + \frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C =$$

$$= e^{2x}\left(\frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{3}{4}\right) + C$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C, \text{ gdzie } a \neq 0$$

15.70

$$\int x\cos xdx = \begin{vmatrix} u = x & dv = \cos xdx \\ du = dx & v = \sin x\end{vmatrix} = x\sin x - \int \sin xdx = x\sin x + \cos x + C$$

15.71

$$\int x^2\cos xdx = \begin{vmatrix} u = x^2 & dv = \cos x\mathrm{d}x \\ du = 2x\mathrm{d}x & v = \sin x\end{vmatrix} = x^2\sin x - 2\int x\sin x\mathrm{d}x =$$

$$= \begin{vmatrix} u = x & dv = \sin xdx \\ du = dx & v = -\cos x\end{vmatrix} = x^2\sin x + 2x\cos x - 2\int \cos x\mathrm{d}x =$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

15.72

$$\int x^2\sin 5xdx = \begin{vmatrix} u = x^2 & dv = \sin 5xdx \\ du = 2xdx & v = -\frac{1}{5}\cos 5x\end{vmatrix} = -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\int x\cos 5x =$$

$$= \begin{vmatrix} u = x & dv = \cos 5x \\ du = dx & v = \frac{1}{5}\sin 5x\end{vmatrix} = -\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x - \frac{2}{25}\int \sin 5xdx =$$

$$= -\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x + \frac{2}{125}\cos 5x + C$$

$$\int \cos axdx = \frac{1}{a}\sin ax + C, \text{ gdzie } a \neq 0$$
$$\int \sin axdx = -\frac{1}{a}\cos ax + C, \text{ gdzie } a \neq 0$$

15.73

$$\int e^x\cos x dx = \begin{vmatrix} u=e^x & dv=\cos x dx \\ du=e^x dx & v=\sin x \end{vmatrix} = e^x\sin x - \int e^x \sin x dx + C =$$

$$= \begin{vmatrix} u=e^x & dv=\sin x \\ du=e^x dx & v=-\cos x \end{vmatrix} = e^x\sin x + e^x\cos x - \int e^x\cos x dx + C$$

$$\int e^x\cos x dx = e^x\sin x + e^x\cos x - \int e^x\cos x dx + C$$

$$2\int e^x\cos x dx = e^x(\sin x+\cos x) + C$$

$$\int e^x\cos x dx = \frac{1}{2}e^x(\sin x+\cos x) + C$$

15.74

$$\int e^{-2x}\sin 3x = \begin{vmatrix} u=e^{-2x} & dv=\sin 3x dx \\ du=-2e^{-2x}dx & v=-\frac{1}{3}\cos 3x \end{vmatrix} = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{3}\int e^{-2x}\cos 3x dx + C =$$

$$= \begin{vmatrix} u=e^{-2x} & dv=\cos 3x dx \\ du=-2e^{-2x} & v=\frac{1}{3}\sin 3x \end{vmatrix} = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x - \frac{4}{9}\int e^{-2x}\sin 3x dx + C$$

$$\int e^{-2x}\sin 3x dx = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x - \frac{4}{9}\int e^{-2x}\sin 3x dx + C$$

$$\frac{13}{9}\int e^{-2x}\sin 3x dx = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x + C$$

$$\int e^{-2x}\sin 3x dx = -\frac{3}{13}e^{-2x}\cos 3x - \frac{2}{13}e^{-2x}\sin 3x + C$$

15.75

$$\int e^x\cos(\tfrac{2}{3}x)dx = \begin{vmatrix} u=e^x & dv=\cos(\frac{2}{3}x)dx \\ du=e^x dx & v=\frac{3}{2}\sin(\frac{2}{3}x) \end{vmatrix} = \frac{3}{2}e^x\sin(\tfrac{2}{3}x) - \frac{3}{2}\int e^x\sin(\tfrac{2}{3}x)dx + C =$$

$$= \begin{vmatrix} u=e^x & dv=\sin(\frac{2}{3}x)dx \\ du=e^x & v=-\frac{3}{2}\cos(\frac{2}{3}x) \end{vmatrix} = \frac{3}{2}e^x\sin(\tfrac{2}{3}x) + \frac{9}{4}e^x\cos(\tfrac{2}{3}x) - \frac{9}{4}\int e^x\cos(\tfrac{2}{3}x) + C$$

$$\int e^x\cos(\tfrac{2}{3}x)dx = \frac{3}{2}e^x\sin(\tfrac{2}{3}x) + \frac{9}{4}e^x\cos(\tfrac{2}{3}x) - \frac{9}{4}\int e^x\cos(\tfrac{2}{3}x) + C$$

$$\frac{13}{4}\int e^x\cos(\tfrac{2}{3}x)dx = \frac{3}{2}e^x\sin(\tfrac{2}{3}x) + \frac{9}{4}e^x\cos(\tfrac{2}{3}x) + C$$

$$\int e^x\cos(\tfrac{2}{3}x)dx = \frac{6}{13}e^x\sin(\tfrac{2}{3}x) + \frac{9}{13}e^x\cos(\tfrac{2}{3}x) + C$$

15.76

$$\int \sqrt{x}\ln x dx = \begin{vmatrix} u=\ln x & dv=\sqrt{x}dx \\ du=\frac{dx}{x} & v=\frac{2}{3}x^{\frac{3}{2}} \end{vmatrix} = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}dx + C = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$$

15.77

$$\int (\ln|x|)^3 dx = \begin{vmatrix} u=\ln^3|x| & dv=dx \\ du=\frac{3\ln^2|x|}{x}dx & v=x \end{vmatrix} = x\ln^3|x| - 3\int \ln^2|x|dx + C =$$

$$= \begin{vmatrix} u = \ln^2|x| & dv = dx \\ u = \frac{2\ln|x|}{x}dx & v = x \end{vmatrix} = x\ln^3|x| - 3x\ln^2|x| + 6\int \ln|x|dx + C = \begin{vmatrix} u = \ln|x| & dv = dx \\ du = \frac{dx}{x} & u = x \end{vmatrix} =$$
$$= x\ln^3|x| - 3x\ln^2|x| + 6x\ln|x| - 6\int dx + C = x\ln^3|x| - 3x\ln^2|x| + 6x\ln|x| - 6x + C$$

15.78

$$\int \frac{(\ln|x|)^2}{x^5}dx = \begin{vmatrix} u = \ln^2|x| & dv = x^{-5}dx \\ du = \frac{2\ln|x|}{x}dx & v = -\frac{1}{4}x^{-4} \end{vmatrix} = -\frac{\ln^2|x|}{4x^4} + \frac{1}{2}\int \frac{\ln|x|}{x^5}dx + C =$$
$$= \begin{vmatrix} u = \ln|x| & dv = x^{-5}dx \\ du = \frac{dx}{x} & v = -\frac{1}{4}x^{-4} \end{vmatrix} = -\frac{\ln^2|x|}{4x^4} - \frac{\ln|x|}{8x^4} + \frac{1}{8}\int \frac{dx}{x^5} + C = -\frac{\ln^2|x|}{4x^4} - \frac{\ln|x|}{8x^4} - \frac{1}{32x^4} + C$$

15.79

$$\int \sqrt{x}(\ln|x|)^3dx = \begin{vmatrix} u = \ln^3|x| & dv = \sqrt{x}dx \\ du = \frac{3\ln^2|x|}{x}dx & v = \frac{2}{3}x^{\frac{3}{2}} \end{vmatrix} = \frac{2}{3}x^{\frac{3}{2}}\ln^3|x| - 2\int \sqrt{x}\ln^2|x| + C =$$
$$= \begin{vmatrix} u = \ln^2|x| & dv = \sqrt{x}dx \\ du = \frac{2\ln|x|}{x}dx & v = \frac{2}{3}x^{\frac{3}{2}} \end{vmatrix} = \frac{2}{3}x^{\frac{3}{2}}\ln^3|x| - \frac{4}{3}x^{\frac{3}{2}}\ln^2|x| + \frac{8}{3}\int \sqrt{x}\ln|x|dx + C =$$
$$= \begin{vmatrix} u = \ln|x| & dv = \sqrt{x}dx \\ du = \frac{dx}{x} & v = \frac{2}{3}x^{\frac{3}{2}} \end{vmatrix} = \frac{2}{3}x^{\frac{3}{2}}\ln^3|x| - \frac{4}{3}x^{\frac{3}{2}}\ln^2|x| + \frac{16}{9}x^{\frac{3}{2}}\ln|x| - \frac{16}{9}\int \sqrt{x}dx + C =$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln^3|x| - \frac{4}{3}x^{\frac{3}{2}}\ln^2|x| + \frac{16}{9}x^{\frac{3}{2}}\ln|x| - \frac{32}{27}x^{\frac{3}{2}} + C$$

15.80

$$\int \frac{\ln|x|}{x^4}dx = \begin{vmatrix} u = \ln|x| & dv = x^{-4}dx \\ du = \frac{dx}{x} & v = -\frac{1}{3}x^{-3} \end{vmatrix} = -\frac{\ln|x|}{3x^3} + \frac{1}{3}\int \frac{dx}{x^4} + C = -\frac{\ln|x|}{3x^3} - \frac{1}{9x^3} + C$$

15.81

$$\int \frac{(\ln x)^2}{\sqrt{x}}dx = \begin{vmatrix} t = \sqrt{x} \\ 2dt = \frac{dx}{\sqrt{x}} \end{vmatrix} = 2\int (\ln t^2)^2dt = 8\int \ln^2 tdt = \begin{vmatrix} u = \ln^2 t & dv = dt \\ du = \frac{2\ln t}{t}dt & v = t \end{vmatrix} =$$
$$= 8t\ln^2 t - 16\int \ln tdt = \begin{vmatrix} u = \ln t & dv = dt \\ du = \frac{dt}{t} & v = t \end{vmatrix} = 8t\ln^2 t - 16t\ln t + 16\int dt + C =$$
$$= 8t\ln^2 t - 16t\ln t + 16t + C = 8\sqrt{x}\ln^2(\sqrt{x}) - 16\sqrt{x}\ln(\sqrt{x}) + 16\sqrt{x} + C =$$
$$= 2\sqrt{x}\ln^2 x - 8\sqrt{x}\ln x + 16\sqrt{x} + C$$

15.82

$$\int x^3(\ln x)^2dx = \begin{vmatrix} u = \ln^2 x & dv = x^3dx \\ du = \frac{2\ln x}{x}dx & v = \frac{1}{4}x^4 \end{vmatrix} = \frac{1}{4}x^4\ln^2 x - \frac{1}{2}\int x^3\ln xdx + C =$$
$$= \begin{vmatrix} u = \ln x & dv = x^3dx \\ du = \frac{dx}{x} & v = \frac{1}{4}x^4 \end{vmatrix} = \frac{1}{4}x^4\ln^2 x - \frac{1}{8}x^4\ln x + \frac{1}{8}\int x^3dx + C = \frac{1}{4}x^4\ln^2 x - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + C$$

15.83

$$\int x^n \ln x \; dx, n \neq -1$$
$$= \begin{vmatrix} u = \ln x & dv = x^n dx \\ du = \frac{dx}{x} & v = \frac{1}{n+1}x^{n+1} \end{vmatrix} = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n dx =$$
$$= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$$

16 Całki funkcji wymiernych.

16.26

$$\int (2x+1)^3 dx = \begin{vmatrix} t = 2x+1 \\ \frac{1}{2}dt = dx \end{vmatrix} = \frac{1}{2}\int t^3 dt = \frac{1}{8}t^4 + C = \frac{1}{8}(2x+1)^4 + C$$

16.27

$$\int \frac{dx}{(3x-2)^4} = \begin{vmatrix} t = 3x-2 \\ \frac{1}{3}dt = dx \end{vmatrix} = \frac{1}{3}\int t^{-4} dt = -\frac{1}{9}t^{-3} + C = -\frac{1}{9(3x-2)^3} + C$$

16.28

$$\int \frac{3x-4}{x^2-x-6}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{3x-4}{x^2-x-6} = \frac{3x-4}{(x-3)(x+2)} \equiv \frac{A}{x-3} + \frac{B}{x+2}$$
$$3x-4 \equiv A(x+2) + B(x-3)$$
$$3x-4 \equiv (A+B)x + (2A-3B)$$
$$\begin{cases} A+B = 3 \\ 2A-3B = -4 \end{cases}$$
$$\begin{cases} A = 1 \\ B = 2 \end{cases}$$
$$\ldots = \int \frac{dx}{x-3} + \int \frac{2dx}{x+2} = \ln|x-3| + 2\ln|x+2| + C$$

16.29

$$\int \frac{2x-3}{x^2-3x+3}dx = \ln|x^2-3x+3| + C$$

$$\boxed{\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C}$$

16.30

$$\int \frac{x+13}{x^2-4x-5}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{x+13}{x^2-4x-5} = \frac{x+13}{(x-5)(x+1)} \equiv \frac{A}{x-5} + \frac{B}{x+1}$$

$$x+13 \equiv A(x+1) + B(x-5)$$

$$x+13 \equiv (A+B)x + (A-5B)$$

$$\begin{cases} A+B=1 \\ A-5B=13 \end{cases}$$

$$\begin{cases} A=3 \\ B=-2 \end{cases}$$

$$\ldots = \int \frac{3dx}{x-5} + \int \frac{-2dx}{x+1} = 3\ln|x-5| - 2\ln|x+1| + C$$

16.31

$$\int \frac{2x+6}{2x^2+3x+1}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{2x+6}{2x^2+3x+1} = \frac{2x+6}{(2x+1)(x+1)} \equiv \frac{A}{2x+1} + \frac{B}{x+1}$$

$$2x+6 \equiv A(x+1) + B(2x+1)$$

$$2x+6 \equiv (A+2B)x + (A+B)$$

$$\begin{cases} A+2B=2 \\ A+B=6 \end{cases}$$

$$\begin{cases} A=10 \\ B=-4 \end{cases}$$

$$\ldots = \int \frac{10}{2x+1}dx + \int \frac{-4}{x+1}dx = 5\ln|2x+1| - 4\ln|x+1| + C$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C, \text{ gdzie } a \neq 0$$

16.32

$$\int \frac{6x-13}{x^2-\frac{7}{2}x+\frac{3}{2}}dx = \int \frac{12x-26}{2x^2-7x+3}dx = \int \frac{12x-26}{(2x-1)(x-3)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{12x-26}{(2x-1)(x-3)} \equiv \frac{A}{2x-1} + \frac{B}{x-3}$$
$$12x-26 \equiv A(x-3)+B(2x-1)$$
$$12x-26 \equiv (A+2B)x+(-3A-B)$$
$$\begin{cases} A+2B=12 \\ -3A-B=-26 \end{cases}$$
$$\begin{cases} A=8 \\ B=2 \end{cases}$$
$$\ldots = \int \frac{8}{2x-1}dx + \int \frac{2}{x-3}dx = 4\ln|2x-1| + 2\ln|x-3| + C$$

16.33

$$\int \frac{4x-5}{2x^2-5x+3}dx = \int \frac{(2x^2-5x+3)'}{2x^2-5x+3}dx = \ln|2x^2-5x+3| + C$$

16.34

$$\int \frac{5x+11}{x^2+3x-10}dx = \int \frac{5x+11}{(x+5)(x-2)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{5x+11}{(x+5)(x-2)} \equiv \frac{A}{x+5} + \frac{B}{x-2}$$
$$5x+11 \equiv A(x-2)+B(x+5)$$
$$5x+11 \equiv (A+B)x+(-2A+5B)$$
$$\begin{cases} A+B=5 \\ -2A+5B=11 \end{cases}$$
$$\begin{cases} A=2 \\ B=3 \end{cases}$$
$$\ldots = \int \frac{2}{x+5}dx + \int \frac{3}{x-2}dx = 2\ln|x+5| + 3\ln|x-2| + C$$

16.35

$$\int \frac{\frac{5}{6}x-16}{x^2+3x-18}dx = \int \frac{\frac{5}{6}x-16}{(x+6)(x-3)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{\frac{5}{6}x-16}{(x+6)(x-3)} \equiv \frac{A}{x+6} + \frac{B}{x-3}$$
$$\frac{5}{6}x-16 \equiv A(x-3)+B(x+6)$$

$\frac{5}{6}x - 16 \equiv (A+B)x + (-3A+6B)$

$\begin{cases} A+B = \frac{5}{6} \\ -3A+6B = -16 \end{cases}$

$\begin{cases} A = \frac{7}{3} \\ B = -\frac{3}{2} \end{cases}$

$\ldots = \int \frac{\frac{7}{3}}{x+6}dx + \int \frac{-\frac{3}{2}}{x-3}dx = \frac{7}{3}\ln|x+6| - \frac{3}{2}\ln|x-3| + C$

16.36

$$\int \frac{dx}{x^2+2x-1} = \int \frac{dx}{(x+1)^2-2} = \int \frac{dx}{(x+1+\sqrt{2})(x+1-\sqrt{2})} = \ldots$$

rozkład na ułamki proste:

$\frac{1}{(x+1+\sqrt{2})(x+1-\sqrt{2})} \equiv \frac{A}{x+1+\sqrt{2}} + \frac{B}{x+1-\sqrt{2}}$

$1 \equiv A(x+1-\sqrt{2}) + B(x+1+\sqrt{2})$

$1 \equiv (A+B)x + [A(1-\sqrt{2}) + B(1+\sqrt{2})]$

$\begin{cases} A+B = 0 \\ A(1-\sqrt{2}) + B(1+\sqrt{2}) = 1 \end{cases}$

$\begin{cases} A = -\frac{1}{2\sqrt{2}} \\ B = \frac{1}{2\sqrt{2}} \end{cases}$

$\ldots = \int \frac{-\frac{1}{2\sqrt{2}}}{x+1+\sqrt{2}} + \int \frac{\frac{1}{2\sqrt{2}}}{x+1-\sqrt{2}}$

$-\frac{1}{2\sqrt{2}}\ln|x+1+\sqrt{2}| + \frac{1}{2\sqrt{2}}\ln|x+1-\sqrt{2}| + C = \frac{1}{2\sqrt{2}}\ln\left|\frac{x+1-\sqrt{2}}{x+1+\sqrt{2}}\right| + C$

16.37

$$\int \frac{dx}{6x^2-13x+6} = \int \frac{dx}{(3x-2)(2x-3)} = \ldots$$

rozkład na ułamki proste:

$\frac{1}{(3x-2)(2x-3)} \equiv \frac{A}{3x-2} + \frac{B}{2x-3}$

$1 \equiv A(2x-3) + B(3x-2)$

$1 \equiv (2A+3B) + (-3A-2B)$

$\begin{cases} 2A+3B = 0 \\ -3A-2B = 1 \end{cases}$

$\begin{cases} A = -\frac{3}{5} \\ B = \frac{2}{5} \end{cases}$

$$\ldots = \int \frac{-\frac{3}{5}}{3x-2}dx + \int \frac{\frac{2}{5}}{2x-3}dx = -\frac{1}{5}\ln|3x-2| + \frac{1}{5}\ln|2x-3| + C$$

16.38

$$\int \frac{5+x}{10x+x^2}dx = \int \frac{\frac{1}{2}(10+2x)}{10x+x^2}dx = \frac{1}{2}\ln|10x+x^2| + C$$

16.39

$$\int \frac{7x}{4+5x^2}dx = \int \frac{\frac{7}{10}\cdot 10x}{4+5x^2}dx = \frac{7}{10}\ln|4+5x^2| + C$$

16.40

$$\int \frac{dx}{-5+6x-x^2} = \int \frac{dx}{2^2-(x-3)^2} = \frac{1}{4}\ln\left|\frac{2+(x-3)}{2-(x-3)}\right| + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \text{ dla } a>0 \wedge |x| \neq a$$

16.41

$$\int \frac{dx}{1+x-x^2} = -\int \frac{dx}{x^2-x-1} = -\int \frac{dx}{(x-\frac{1}{2})^2-\frac{5}{4}} = -\int \frac{dx}{(x-\frac{1+\sqrt{5}}{2})(x-\frac{1-\sqrt{5}}{2})} = \ldots$$

rozkład na ułamki proste:

$$\frac{1}{(x-\frac{1+\sqrt{5}}{2})(x-\frac{1-\sqrt{5}}{2})} \equiv \frac{A}{x-\frac{1+\sqrt{5}}{2}} + \frac{B}{x-\frac{1-\sqrt{5}}{2}}$$

$$1 \equiv A(x-\frac{1-\sqrt{5}}{2}) + B(x-\frac{1+\sqrt{5}}{2})$$

$$1 \equiv (A+B)x + \left(-A\cdot\frac{1-\sqrt{5}}{2} - B\cdot\frac{1+\sqrt{5}}{2}\right)$$

$$\begin{cases} A+B=0 \\ -A\cdot\frac{1-\sqrt{5}}{2} - B\cdot\frac{1+\sqrt{5}}{2} = 1 \end{cases}$$

$$\begin{cases} A = \frac{1}{\sqrt{5}} \\ B = -\frac{1}{\sqrt{5}} \end{cases}$$

$$\ldots = -\left[\int \frac{\frac{1}{\sqrt{5}}}{x-\frac{1+\sqrt{5}}{2}}dx + \int \frac{-\frac{1}{\sqrt{5}}}{x-\frac{1-\sqrt{5}}{2}}dx\right] = \frac{\ln|x-\frac{1-\sqrt{5}}{2}| - \ln|x-\frac{1+\sqrt{5}}{2}|}{\sqrt{5}} + C$$

16.42

$$\int \frac{dx}{2x-3x^2} = \int \frac{dx}{x(2-3x)} = \ldots$$

rozkład na ułamki proste:

$$\frac{1}{x(2-3x)} \equiv \frac{A}{x} + \frac{B}{2-3x}$$
$$1 \equiv A(2-3x) + Bx$$
$$1 \equiv (-3A+B)x + 2A$$
$$\begin{cases} -3A+B=0 \\ 2A=1 \end{cases}$$
$$\begin{cases} A=\frac{1}{2} \\ B=\frac{3}{2} \end{cases}$$
$$\ldots = \int \frac{\frac{1}{2}}{x}dx + \int \frac{\frac{3}{2}}{2-3x}dx = \frac{1}{2}\ln|x| - \frac{1}{2}\ln|2-3x| + C$$

16.43

$$\int \frac{3x+2}{x^2-x-2}dx = \int \frac{3x+2}{(x+1)(x-2)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{3x+2}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$
$$3x+2 \equiv A(x-2) + B(x+1)$$
$$3x+2 \equiv (A+B)x + (-2A+B)$$
$$\begin{cases} A+B=3 \\ -2A+B=2 \end{cases}$$
$$\begin{cases} A=\frac{1}{3} \\ B=\frac{8}{3} \end{cases}$$
$$\ldots = \int \frac{\frac{1}{3}}{x+1}dx + \int \frac{\frac{8}{3}}{x-2}dx = \frac{1}{3}\ln|x+1| + \frac{8}{3}\ln|x-2| + C$$

16.44

$$\int \frac{2x-1}{x^2-6x+9}dx = \int \frac{2x-6+5}{x^2-6x+9}dx = \int \frac{(x^2-6x+9)'}{x^2-6x+9}dx + \int \frac{5}{(x-3)^2}dx$$
$$\ln|x^2-6x+9| - \frac{5}{x-3} + C$$

16.45

$$\int \frac{x-1}{4x^2-4x+1}dx = \int \frac{\frac{1}{8}(4x^2-4x+1)' - \frac{1}{2}}{4x^2-4x+1}dx = \frac{1}{8}\ln|4x^2-4x+1| - \frac{1}{2}\int \frac{dx}{(2x-1)^2}$$
$$\frac{1}{8}\ln|(2x-1)^2| - \frac{1}{2}\cdot\frac{-1}{2(2x-1)} + C = \frac{1}{4}\ln|2x-1| + \frac{1}{4(2x-1)} + C$$

16.46

$$\int \frac{2x-13}{(x-5)^2}dx = \int \frac{2(x-5)-3}{(x-5)^2}dx = \int \frac{2}{x-5}dx - \int \frac{3}{(x-5)^2}$$

$$2\ln|x-5| + \frac{3}{x-5} + C$$

16.47

$$\int \frac{3x+1}{(x+2)^2}dx = \int \frac{3(x+2)-5}{(x+2)^2}dx = \int \frac{3}{x+2}dx - \int \frac{5}{(x+2)^2}dx$$

$$3\ln|x+2| + \frac{5}{x+2} + C$$

16.48

$$\int \frac{dx}{2x^2-2x+5} = \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{3}{2})^2} = \frac{1}{3}\arctan\left(\frac{2x-1}{3}\right) + C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \text{ gdzie } a \neq 0$$

16.49

$$\int \frac{dx}{3x^2+2x+1} = \frac{1}{3}\int \frac{dx}{(x+\frac{1}{3})^2+(\frac{\sqrt{2}}{3})^2} = \frac{1}{\sqrt{2}}\arctan\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

16.50

$$\int \frac{dx}{13-6x+x^2} = \int \frac{dx}{(x-3)^2+2^2} = \frac{1}{2}\arctan\left(\frac{x-3}{2}\right) + C$$

16.51

$$\int \frac{3dx}{9x^2-6x+2} = \int \frac{3dx}{(3x-1)^2+1} = \left| \begin{matrix} t = 3x-1 \\ dt = 3dx \end{matrix} \right| = \int \frac{dt}{t^2+1}$$

$$\arctan t + C = \arctan(3x-1) + C$$

16.52

$$\int \frac{x+1}{x^2-x+1}dx = \int \frac{\frac{1}{2}(x^2-x+1)' + \frac{3}{2}}{x^2-x+1}dx = \frac{1}{2}\ln|x^2-x+1| + \frac{3}{2}\int \frac{dx}{(x-\frac{1}{2})^2-(\frac{\sqrt{3}}{2})^2}$$

$$\frac{1}{2}\ln|x^2-x+1| + \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

16.53

$$\int \frac{4x-1}{2x^2-2x+1}dx = \int \frac{(2x^2-2x+1)'+1}{2x^2-2x+1}dx = \ln|2x^2-2x+1| + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{1}{2})^2}$$
$$\ln|2x^2-2x+1| + \arctan(2x-1) + C$$

16.54

$$\int \frac{2x-1}{x^2-2x+5}dx = \int \frac{(x^2-2x+5)'+1}{x^2-2x+5}dx = \ln|x^2-2x+5| + \int \frac{dx}{(x-1)^2+2^2}$$
$$\ln|x^2-2x+5| + \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$$

16.55

$$\int \frac{2x-10}{x^2-2x+10}dx = \int \frac{(x^2-2x+10)'-8}{x^2-2x+10}dx = \ln|x^2-2x+10| - 8\int \frac{dx}{(x-1)^2+3^2}$$
$$\ln|x^2-2x+10| - \frac{8}{3}\arctan\left(\frac{x-1}{3}\right) + C$$

16.56

$$\int \frac{2x-20}{x^2-8x+25}dx = \int \frac{(x^2-8x+25)'-12}{x^2-8x+25}dx = \ln|x^2-8x+25| - 12\int \frac{dx}{(x-4)+3^2}$$
$$\ln|x^2-8x+25| - 4\arctan\left(\frac{x-4}{3}\right) + C$$

16.57

$$\int \frac{3x+4}{x^2+4x+8}dx = \int \frac{\frac{3}{2}(x^2+4x+8)'-2}{x^2+4x+8}dx = \frac{3}{2}\ln|x^2+4x+8| - 2\int \frac{dx}{(x+2)^2+2^2}$$
$$\frac{3}{2}\ln|x^2+4x+8| - \arctan\left(\frac{x+2}{2}\right) + C$$

16.58

$$\int \frac{x+6}{x^2-3}dx = \int \frac{\frac{1}{2}(x^2-3)'+6}{x^2-3}dx = \frac{1}{2}\ln|x^2-3| + 6\int \frac{dx}{x^2-3}$$
$$\frac{1}{2}\ln|x^2-3| - 6\int \frac{dx}{3-x^2} = \frac{1}{2}\ln|x^2-3| - \sqrt{3}\ln\left|\frac{\sqrt{3}+x}{\sqrt{3}-x}\right| + C$$

16.59

$$\int \frac{x+6}{x^2+3}dx = \int \frac{\frac{1}{2}(x^2+3)'+6}{x^2+3}dx = \frac{1}{2}\ln|x^2+3| + 2\sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

16.60

$$\int \frac{6x}{x^2+4x+13}dx = \int \frac{3(x^2+4x+13)'-12}{x^2+4x+13} = 3\ln|x^2+4x+13| - 12\int \frac{dx}{(x+2)^2+3^2}$$
$$3\ln|x^2+4x+13| - 4\arctan\left(\frac{x+2}{3}\right) + C$$

16.61

$$\int \frac{10x-44}{x^2-4x+20}dx = \int \frac{5(x^2-4x+20)'-24}{x^2-4x+20}dx = 5\ln|x^2-4x+20| - 24\int \frac{dx}{(x-2)^2+4^2}$$
$$5\ln|x^2-4x+20| - 6\arctan\left(\frac{x-2}{4}\right) + C$$

16.62

$$\int \frac{4x-5}{x^2-6x+10}dx = \int \frac{2(x^2-6x+10)'+7}{x^2-6x+10}dx = 2\ln|x^2-6x+10| + 7\int \frac{dx}{(x-3)^2+1}$$
$$2\ln|x^2-6x+10| + 7\arctan(x-3) + C$$

16.63

$$\int \frac{5x}{2+3x}dx = \int \frac{\frac{5}{3}(3x+2)-\frac{10}{3}}{3x+2}dx = \frac{5}{3}x - \frac{10}{9}\ln|3x+2| + C$$

16.64

$$\int \frac{x^2}{5x^2+12}dx = \frac{1}{5}\int \frac{x^2+\frac{12}{5}-\frac{12}{5}}{x^2+\frac{12}{5}} = \frac{1}{5}x - \frac{12}{25}\int \frac{dx}{x^2+(2\sqrt{\frac{3}{5}})^2}$$
$$\frac{1}{5}x - \frac{6}{25}\sqrt{\frac{5}{3}}\arctan\left(\frac{x}{2}\sqrt{\frac{5}{3}}\right) + C$$

16.65

$$\int \frac{2x^2+7x+20}{x^2+6x+25}dx = \int \frac{2(x^2+6x+25)-5x-30}{x^2+6x+25}dx = 2x - \int \frac{\frac{5}{2}(x^2+6x+25)'+15}{x^2+6x+25}dx =$$
$$= 2x - \frac{5}{2}\ln|x^2+6x+25| - 15\int \frac{dx}{(x+3)^2+4^2} =$$
$$= 2x - \frac{5}{2}\ln|x^2+6x+25| - \frac{15}{4}\arctan\left(\frac{x+3}{4}\right) + C$$

16.66

$$\int \frac{7x^2+7x-176}{x^3-9x^2+6x+56}dx = \int \frac{7x^2+7x-176}{(x+2)(x-4)(x-7)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{7x^2+7x-176}{(x+2)(x-4)(x-7)} \equiv \frac{A}{x+2}+\frac{B}{x-4}+\frac{C}{x-7}$$
$$7x^2+7x-176 \equiv A(x-4)(x-7)+B(x+2)(x-7)+C(x+2)(x-4)$$
$$7x^2+7x-176 \equiv (A+B+C)x^2+(-11A-5B-2C)x+(28A-14B-8C)$$
$$\begin{cases} A+B+C=7 \\ -11A-5B-2C=7 \\ 28A-14B-8C=-176 \end{cases}$$
$$\begin{cases} A=-3 \\ B=2 \\ C=8 \end{cases}$$
$$\ldots = \int \frac{-3}{x+2}dx + \int \frac{2}{x-4}dx + \int \frac{8}{x-7}dx$$
$$-3\ln|x+2| + 2\ln|x-4| + 8\ln|x-7| + C$$

16.67

$$\int \frac{x^3-4x^2+1}{(x-2)^4}dx = \ldots$$

rozkład na ułamki proste:

$$\int \frac{x^3-4x^2+1}{(x-2)^4} \equiv \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{D}{(x-2)^4}$$
$$x^3-4x^2+1 \equiv A(x-2)^3+B(x-2)^2+C(x-2)+D$$
$$x^3-4x^2+1 \equiv Ax^3+(-6A+B)x^2+(12Ax-4B+C)x+(-8A+4B-2C+D)$$
$$\begin{cases} A=1 \\ -6A+B=-4 \\ 12A-4B+C=0 \\ -8A+4B-2C+D=1 \end{cases}$$
$$\begin{cases} A=1 \\ B=2 \\ C=-4 \\ D=-7 \end{cases}$$
$$\ldots = \int \frac{dx}{x-2} + \int \frac{2}{(x-2)^2}dx + \int \frac{-4}{(x-2)^3}dx + \int \frac{-7}{(x-2)^4}dx =$$
$$= \ln|x-2| - \frac{2}{x-2} + \frac{2}{(x-2)^2} + \frac{7}{3(x-2)^3} + C$$

16.68

$$\int \frac{3x^2-5x+2}{x^3-2x^2+3x-6} = \int \frac{3x^2-5x+2}{(x^2+3)(x-2)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{3x^2-5x+2}{(x^2+3)(x-2)} \equiv \frac{Ax+B}{x^2+3} + \frac{C}{x-2}$$
$$3x^2-5x+2 \equiv (Ax+B)(x-2)+C(x^2+3)$$
$$3x^2-5x+2 \equiv (A+C)x^2+(-2A+B)x+(-2B+3C)$$
$$\begin{cases} A+C=3 \\ -2A+B=-5 \\ -2B+3C=2 \end{cases}$$
$$\begin{cases} A=\frac{17}{7} \\ B=-\frac{1}{7} \\ C=\frac{4}{7} \end{cases}$$
$$\ldots = \int \frac{\frac{17}{7}x-\frac{1}{7}}{x^2+3}dx + \int \frac{\frac{4}{7}}{x-2} = \frac{17}{14}\ln|x^2+3| - \frac{1}{7}\int\frac{dx}{x^2+3} + \frac{4}{7}\ln|x-2| =$$
$$= \frac{17}{14}\ln|x^2+3| - \frac{1}{7\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + \frac{4}{7}\ln|x-2| + C$$

16.69

$$\int \frac{2x+1}{(x^2+1)^2}dx = \underbrace{\int \frac{2x}{(x^2+1)^2}dx}_{1} + \underbrace{\int \frac{dx}{(x^2+1)^2}}_{2} = \ldots$$

1)

$$\int \frac{2x}{(x^2+1)^2}dx = \begin{vmatrix} t=x^2+1 \\ dt=2xdx \end{vmatrix} = \int t^{-2}dt = -\frac{1}{t} + C = -\frac{1}{x^2+1} + C$$

2)

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{x^2+1-x^2}{(x^2+1)^2}dx = \int \frac{dx}{x^2+1} - \int \frac{x^2}{(x^2+1)^2}dx =$$
$$= \arctan x - \begin{vmatrix} u=x & dv=\frac{xdx}{(x^2+1)^2} \\ du=dx & v=-\frac{1}{2(x^2+1)} \end{vmatrix} = \arctan x + \frac{x}{2(x^2+1)} - \frac{1}{2}\int\frac{dx}{x^2+1} =$$
$$= \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)} + C$$
$$\ldots = -\frac{1}{x^2+1} + \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)} + C = \frac{x-2}{2(x^2+1)} + \frac{1}{2}\arctan x + C$$

16.70

$$\int \frac{x^3+2x-6}{x^2-x-2}dx = \int \frac{x(x^2-x-2)+x^2+4x-6}{x^2-x-2}dx = \frac{1}{2}x^2 + \int \frac{(x^2-x-2)+5x-4}{x^2-x-2}dx =$$
$$= \frac{1}{2}x^2 + x + \int \frac{5x-4}{x^2-x-2}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{5x-4}{x^2-x-2} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

$5x-4\equiv A(x-2)+B(x+1)$
$5x-4\equiv (A+B)x+(-2A+B)$
$\begin{cases}A+B=5\\-2A+B=-4\end{cases}$
$\begin{cases}A=3\\B=2\end{cases}$
$\ldots=\frac{1}{2}x^2+x+\int\frac{3dx}{x+1}+\int\frac{2dx}{x-2}=\frac{1}{2}x^2+x+3\ln|x+1|+2\ln|x-2|+C$

16.71

$$\int\frac{2x^3-19x^2+58x-42}{x^2-8x+16}dx=\int\frac{2x(x^2-8x+16)-3x^2+26x-42}{x^2-8x+16}dx=$$
$$=x^2+\int\frac{-3(x^2-8x+16)+2x+6}{x^2-8x+16}dx=x^2-3x+\int\frac{(x^2-8x+16)'+14}{(x-4)^2}dx=$$
$$=x^2-3x+2\ln|x-4|-\frac{14}{x-4}+C$$

16.72

$$\int\frac{x^4}{x^2+1}dx=\int\frac{(x^2-1)(x^2+1)+1}{x^2+1}dx=\int(x^2-1)dx+\int\frac{dx}{x^2+1}=$$
$$=\frac{1}{3}x^3-x+\arctan x+C$$

16.73

$$\int\frac{72x^6}{3x^2+2}dx=\int\frac{24x^4(3x^2+2)-48x^4}{3x^2+2}dx=\int 24x^4dx-\int\frac{16x^2(3x^2+2)-32x^2}{3x^2+2}=$$
$$=\frac{24}{5}x^5-\int 16x^2dx+\int\frac{\frac{32}{3}(3x^2+2)-\frac{64}{3}}{3x^2+2}dx=\frac{24}{5}x^5-\frac{16}{3}x^3+\frac{32}{3}x-\frac{64}{9}\int\frac{dx}{x^2+\frac{2}{3}}=$$
$$=\frac{24}{5}x^5-\frac{16}{3}x^3+\frac{32}{3}x-\frac{32}{3}\sqrt{\frac{2}{3}}\arctan\left(x\sqrt{\frac{3}{2}}\right)+C$$

16.74

$$\int\frac{2x^4-10x^3+21x^2-20x+5}{x^2-3x+2}dx=\int\frac{2x^2(x^2-3x+2)-4x^3+17x^2-20x+5}{x^2-3x+2}dx=$$
$$=\frac{2}{3}x^3+\int\frac{-4x(x^2-3x+2)+5x^2-12x+5}{x^2-3x+2}dx=$$
$$=\frac{2}{3}x^3-2x^2+\int\frac{5(x^2-3x+2)+3x-5}{x^2-3x+2}dx=\frac{2}{3}x^3-2x^2+5x+\int\frac{3x-5}{x^2-3x+2}dx=\ldots$$

rozkład na ułamki proste:

$\frac{3x-5}{x^2-3x+2}\equiv\frac{A}{x-1}+\frac{B}{x-2}$
$3x-5\equiv A(x-2)+B(x-1)$

$3x - 5 \equiv (A+B)x + (-2A - B)$

$\begin{cases} A + B = 3 \\ -2A - B = -5 \end{cases}$

$\begin{cases} A = 2 \\ B = 1 \end{cases}$

$\ldots = \frac{2}{3}x^3 - 2x^2 + 5x + \int \frac{2dx}{x-1} + \int \frac{dx}{x-2} =$

$= \frac{2}{3}x^3 - 2x^2 + 5x + 2\ln|x-1| + \ln|x-2| + C$

16.75

$\int \frac{x^2+5x+41}{(x+3)(x-1)(x-\frac{1}{2})} dx = \ldots$

rozkład na ułamki proste:

$\frac{x^2+5x+41}{(x+3)(x-1)(x-\frac{1}{2})} \equiv \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{x-\frac{1}{2}}$

$x^2+5x+41 \equiv A(x-1)(x-\frac{1}{2}) + B(x+3)(x-\frac{1}{2}) + C(x+3)(x-1)$

$x^2+5x+41 \equiv (A+B+C)x^2 + (-\frac{3}{2}A + \frac{5}{2}B + 2C)x + (\frac{1}{2}A - \frac{3}{2}B - 3C)$

$\begin{cases} A + B + C = 1 \\ -3A + 5B + 4C = 10 \\ A - 3B - 6C = 82 \end{cases}$

$\begin{cases} A = \frac{5}{2} \\ B = \frac{47}{2} \\ C = -25 \end{cases}$

$\ldots = \int \frac{\frac{5}{2}}{x+3} dx + \int \frac{\frac{47}{2}}{x-1} dx + \int \frac{-25}{x-\frac{1}{2}} dx =$

$= \frac{5}{2}\ln|x+3| + \frac{47}{2}\ln|x-1| - 25\ln|x-\frac{1}{2}| + C$

16.76

$\int \frac{17x^2-x-26}{(x^2-1)(x^2-4)} dx = \ldots$

rozkład na ułamki proste:

$\frac{17x^2-x-26}{(x^2-1)(x^2-4)} \equiv \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+2} + \frac{D}{x-2}$

$17x^2 - x - 26 \equiv A(x-1)(x^2-4) + B(x+1)(x^2-4) + C(x^2-1)(x-2) + D(x^2-1)(x+2)$

$\begin{cases} A + B + C + D = 0 \\ -A + B - 2C + 2D = 17 \\ -4A - 4B - C - D = -1 \\ 4A - 4B + 2C - 2D = -26 \end{cases}$

$$\begin{cases} A = -\frac{4}{3} \\ B = \frac{5}{3} \\ C = -\frac{11}{3} \\ D = \frac{10}{3} \end{cases}$$

$$\ldots = \int \frac{-\frac{4}{3}}{x+1}dx + \int \frac{\frac{5}{3}}{x-1}dx + \int \frac{-\frac{11}{3}}{x+2}dx + \int \frac{\frac{10}{3}}{x-2}dx =$$
$$= -\frac{4}{3}\ln|x+1| + \frac{5}{3}\ln|x-1| - \frac{11}{3}\ln|x+2| + \frac{10}{3}\ln|x-2| + C$$

16.77

$$\int \frac{2x}{(x^2+1)(x^2+3)}dx = \ldots$$
rozkład na ułamki proste:
$$\frac{2x}{(x^2+1)(x^2+3)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$
$$2x \equiv (Ax+B)(x^2+3) + (Cx+D)(x^2+1)$$
$$2x \equiv (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D)$$
$$\begin{cases} A+C=0 \\ B+D=0 \\ 3A+C=2 \\ 3B+D=0 \end{cases}$$
$$\begin{cases} A=1 \\ B=0 \\ C=-1 \\ D=0 \end{cases}$$
$$\ldots = \int \frac{xdx}{x^2+1} - \int \frac{xdx}{x^2+3} = \frac{1}{2}\ln|x^2+1| - \frac{1}{2}\ln|x^2+3| + C$$

16.78

$$\int \frac{10x^3+110x+400}{(x^2-4x+29)(x^2-2x+5)}dx = \ldots$$
rozkład na ułamki proste:
$$\frac{10x^3+110x+400}{(x^2-4x+29)(x^2-2x+5)} \equiv \frac{Ax+B}{x^2-4x+29} + \frac{Cx+D}{x^2-2x+5}$$
$$10x^3+110x+400 \equiv (Ax+B)(x^2-2x+5) + (Cx+D)(x^2-4x+29)$$
$$\begin{cases} A+C=10 \\ -2A+B-4C+D=0 \\ 5A-2B+29C-4D=110 \\ 5B+29D=400 \end{cases}$$

$$\begin{cases} A = 4 \\ B = 22 \\ C = 6 \\ D = 10 \end{cases}$$

$$\ldots = \int \frac{4x+22}{x^2-4x+29}dx + \int \frac{6x+10}{x^2-2x+5}dx =$$

$$= \int \frac{2(x^2-4x+29)'+30}{(x-2)^2+5^2}dx + \int \frac{3(x^2-2x+5)'+16}{(x-1)^2+2^2}dx =$$

$$= 2\ln|x^2-4x+29| + 6\arctan\left(\frac{x-2}{5}\right) + 3\ln|x^2-2x+5| + 8\arctan\left(\frac{x-1}{2}\right) + C$$

16.79

$$\int \frac{4x^3-2x^2+6x-13}{x^4+3x^2-4}dx = \int \frac{4x^3-2x^2+6x-13}{(x^2+4)(x^2-1)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{4x^3-2x^2+6x-13}{(x^2+4)(x^2-1)} \equiv \frac{Ax+B}{x^2+4} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$4x^3-2x^2+6x-13 \equiv (Ax+B)(x^2-1) + C(x^2+4)(x-1) + D(x^2+4)(x+1)$$

$$4x^3-2x^2+6x-13 \equiv (A+C+D)x^3 + (B-C+D)x^2 + (-A+4C+4D)x + (-B-4C+4D)$$

$$\begin{cases} A+C+D = 4 \\ B-C+D = -2 \\ -A+4C+4D = 6 \\ -B-4C+4D = -13 \end{cases}$$

$\ldots$

$$\begin{cases} A = 2 \\ B = 1 \\ C = \frac{5}{2} \\ D = -\frac{1}{2} \end{cases}$$

$$\ldots = \int \frac{2x+1}{x^2+4} + \int \frac{\frac{5}{2}}{x+1} + \int \frac{-\frac{1}{2}}{x-1} =$$

$$= \ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{5}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$$

16.80

$$\int \frac{10x^3+40x^2+40x+6}{x^4+6x^3+11x^2+6x}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{10x^3+40x^2+40x+6}{x^4+6x^3+11x^2+6x} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3}$$

$$10x^3+40x^2+40x+6 \equiv A(x+1)(x+2)(x+3) + Bx(x+2)(x+3) + Cx(x+1)(x+3) + Dx(x+1)(x+2)$$

$10x^3+40x^2+40x+6\equiv(A+B+C+D)x^3+(6A+5B+4C+3D)x^2+(11A+6B+3C+2D)x+6A$

$$\begin{cases} A+B+C+D=10 \\ 6A+5B+4C+3D=40 \\ 11A+6B+3C+2D=40 \\ 6A=6 \end{cases}$$

...

$$\begin{cases} A=1 \\ B=2 \\ C=3 \\ D=4 \end{cases}$$

$\ldots=\int\frac{dx}{x}+\int\frac{2dx}{x+1}+\int\frac{3dx}{x+2}+\int\frac{4dx}{x+3}=$
$=\ln|x|+2\ln|x+1|+3\ln|x+2|+4\ln|x+3|+C$

16.81

$$\int\frac{6x^3+4x+1}{x^4+x^2}dx=\ldots$$

rozkład na ułamki proste:

$\frac{6x^3+4x+1}{x^4+x^2}\equiv\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$
$6x^3+4x+1\equiv A(x^3+x)+B(x^2+1)+(Cx+D)x^2$
$6x^3+4x+1\equiv(A+C)x^3+(B+D)x^2+Ax+B$

$$\begin{cases} A+C=6 \\ B+D=0 \\ A=4 \\ B=1 \end{cases}$$

$$\begin{cases} A=4 \\ B=1 \\ C=2 \\ D=-1 \end{cases}$$

$\ldots=\int\frac{4dx}{x}+\int\frac{dx}{x^2}+\int\frac{2x-1}{x^2+1}dx=4\ln|x|-\frac{1}{x}+\ln|x^2+1|-\arctan x+C$

16.82

$$\int\frac{dx}{x^3-a^2x}=\ldots$$

dla

$a=0\rightarrow\int\frac{dx}{x^3}=-\frac{1}{2x^2}+C$

dla

$a \neq 0$

rozkład na ułamki proste:

$$\frac{1}{x^3 - a^2x} \equiv \frac{A}{x} + \frac{Bx + C}{x^2 - a^2}$$
$$1 \equiv A(x^2 - a^2) + (Bx + C)x$$
$$1 \equiv (A + B)x^2 + Cx - a^2A$$
$$\begin{cases} A + B = 0 \\ C = 0 \\ -a^2A = 1 \end{cases}$$
$$\begin{cases} A = -\frac{1}{a^2} \\ B = \frac{1}{a^2} \\ C = 0 \end{cases}$$
$$\ldots = \int \frac{-\frac{1}{a^2}}{x} + \int \frac{\frac{1}{a^2}x}{x^2 - a^2} = -\frac{1}{a^2} \ln|x| + \frac{1}{2a^2} \ln|x^2 - a^2| + C$$

16.83

$$\int \frac{dx}{x^3 + x^2 + x} = \ldots$$

rozkład na ułamki proste:

$$\frac{1}{x^3 + x^2 + x} \equiv \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}$$
$$1 \equiv A(x^2 + x + 1) + (Bx + C)x$$
$$1 \equiv (A + B)x^2 + (A + C)x + A$$
$$\begin{cases} A + B = 0 \\ A + C = 0 \\ A = 1 \end{cases}$$
$$\begin{cases} A = 1 \\ B = -1 \\ C = -1 \end{cases}$$
$$\ldots = \int \frac{dx}{x} + \int \frac{-x - 1}{x^2 + x + 1} = \ln|x| + \int \frac{-\frac{1}{2}(x^2 + x + 1)' - \frac{1}{2}}{(x + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} dx =$$
$$= \ln|x| - \frac{1}{2} \ln|x^2 + x + 1| - \frac{1}{\sqrt{3}} \arctan\left(\frac{2x + 1}{\sqrt{3}}\right) + C$$

16.84

$$\int \frac{dx}{x^4 + x^2 + 1} = \ldots$$

rozkład na ułamki proste:

$$\frac{1}{x^4+x^2+1} \equiv \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+x+1}$$
$$1 \equiv (Ax+B)(x^2+x+1)(Cx+D)(x^2-x+1)$$
$$1 \equiv (A+C)x^3 + (A+B-C+D)x^2 + (A+B+C-D)x + (B+D)$$
$$\begin{cases} A+C=0 \\ A+B-C+D=0 \\ A+B+C-D=0 \\ B+D=0 \end{cases}$$
...
$$\begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \\ C=\frac{1}{2} \\ D=\frac{1}{2} \end{cases}$$
$$\ldots = \int \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2-x+1}dx + \int \frac{\frac{1}{2}x+\frac{1}{2}}{x^2+x+1} = \int \frac{-\frac{1}{4}(x^2-x+1)'+\frac{1}{4}}{x^2-x+1}dx + \int \frac{\frac{1}{4}(x^2+x+1)'+\frac{1}{4}}{x^2+x+1}dx =$$
$$= -\frac{1}{4}\ln|x^2-x+1| + \int \frac{\frac{1}{4}}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{4}\ln|x^2+x+1| + \int \frac{\frac{1}{4}}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dx =$$
$$= \frac{1}{4}\ln\left|\frac{x^2+x+1}{x^2-x+1}\right| + \frac{1}{2\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

16.85

$$\int \frac{5x^3+3x^2+12x-12}{x^4-16}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{5x^3+3x^2+12x-12}{x^4-16} \equiv \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$
$$5x^3+3x^2+12x-12 \equiv A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x^2-4)$$
$$5x^3+3x^2+12x-12 \equiv (A+B+C)x^3 + (2A-2B+D)x^2 + (4A+4B-4C)x + (8A-8B-4D)$$
$$\begin{cases} A+B+C=5 \\ 2A-2B+D=3 \\ 4A+4B-4C=12 \\ 8A-8B-4D=-12 \end{cases}$$
$$\begin{cases} A=2 \\ B=2 \\ C=1 \\ D=3 \end{cases}$$
$$\ldots = \int \frac{2dx}{x-2} + \int \frac{2dx}{x+2} + \int \frac{x+3}{x^2+4}dx =$$
$$= 2\ln|x-2| + 2\ln|x+2| + \frac{1}{2}\ln|x^2+4| + \frac{3}{2}\arctan\frac{x}{2} + C$$

16.86

$$\int \frac{15x^2+66x+21}{(x-1)(x^2+4x+29)}dx = \dots$$

rozkład na ułamki proste:

$$\frac{15x^2+66x+21}{(x-1)(x^2+4x+29)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+29}$$
$$15x^2+66x+21 \equiv A(x^2+4x+29)+(Bx+C)(x-1)$$
$$15x^2+66x+21 \equiv (A+B)x^2+(4A-B+C)x+(29A-C)$$
$$\begin{cases} A+B=15 \\ 4A-B+C=66 \\ 29A-C=21 \end{cases}$$
$$\begin{cases} A=3 \\ B=12 \\ C=66 \end{cases}$$
$$\dots = \int \frac{3dx}{x-1} + \int \frac{12x+66}{x^2+4x+29}dx = 3\ln|x-1| + \int \frac{6(x^2+4x+29)'+42}{(x+2)^2+5^2}dx =$$
$$= 3\ln|x-1| + 6\ln|x^2+6x+29| + \frac{42}{5}\arctan\left(\frac{x+2}{5}\right) + C$$

16.87

$$\int \frac{4x^3+9x^2+4x+1}{x^4+3x^3+3x^2+x}dx = \int \frac{(x^4+3x^3+3x^2+x)'-2x}{x^4+3x^3+3x^2+x}dx =$$
$$= \ln|x^4+3x^3+3x^2+x| - \int \frac{2x}{x(x+1)^3}dx = \ln|x^4+3x^3+3x^2+x| - \int \frac{2dx}{(x+1)^3} =$$
$$= \ln|x^4+3x^3+3x^2+x| + \frac{1}{(x+1)^2} + C$$

16.88

$$\int \frac{dx}{x^3(x-1)^2(x+1)} = \dots$$

rozkład na ułamki proste:

$$\frac{1}{x^3(x-1)^2(x+1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{x+1}$$
$$1 \equiv Ax^2(x-1)^2(x+1) + Bx(x-1)^2(x+1) + C(x-1)^2(x+1) +$$
$$+ Dx^3(x-1)(x+1) + Ex^3(x+1) + Fx^3(x-1)^2$$
$$1 \equiv (A+D+F)x^5 + (-A+B+E-2F)x^4 + (-A-B+C-D+E+F)x^3 +$$
$$+ (A-B-C)x^2 + (B-C)x + C$$

$$\begin{cases} A+D+F=0 \\ -A+B+E-2F=0 \\ -A-B+C-D+E+F=0 \\ A-B-C=0 \\ B-C=0 \\ C=1 \end{cases}$$

...

$$\begin{cases} A=2 \\ B=1 \\ C=1 \\ D=-\frac{7}{4} \\ E=\frac{1}{2} \\ F=-\frac{1}{4} \end{cases}$$

$$\ldots = \int \frac{2dx}{x} + \int \frac{dx}{x^2} + \int \frac{dx}{x^3} + \int \frac{-\frac{7}{4}dx}{x-1} + \int \frac{\frac{1}{2}dx}{(x-1)^2} + \int \frac{-\frac{1}{4}dx}{x+1} =$$
$$= 2\ln|x| - \frac{1}{x} - \frac{1}{2x^2} - \frac{7}{4}\ln|x-1| - \frac{1}{2(x-1)} - \frac{1}{4}\ln|x+1| + C$$

16.89

$$\int \frac{dx}{(x^2+x+1)^2} = \int \frac{dx}{[(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2]^2} = \left(\frac{4}{3}\right)^2 \int \frac{dx}{\left[\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1\right]^2} =$$

$$= \frac{16}{9}\int \frac{dx}{\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]^2} = \left| \begin{array}{l} t=\frac{2x+1}{\sqrt{3}} \\ dt=\frac{2}{\sqrt{3}}dx \\ \frac{\sqrt{3}}{2}dt=dx \end{array} \right| = \frac{8}{3\sqrt{3}}\int \frac{dt}{(t^2+1)^2} = \ldots$$

korzystając z wyliczonej całki w zadaniu (16.69) :

$$\ldots = \frac{8}{3\sqrt{3}}\left(\frac{1}{2}\arctan t + \frac{t}{2(t^2+1)} + C\right) = \frac{4}{3\sqrt{3}}\arctan t + \frac{4t}{3\sqrt{3}(t^2+1)} + C =$$
$$= \frac{4}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{2x+1}{3(x^2+x+1)} + C$$

Wzór rekurencyjny:

$$\mathcal{I}_n = \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\mathcal{I}_{n-1},\ \text{gdzie}\ \mathcal{I}_n = \int \frac{dx}{(x^2+1)^n}$$

16.90

$$\int \frac{3x^2-17x+21}{(x-2)^3}dx = \ldots$$

$[(x-2)^3]' = 3(x-2)^2 = 3x^2 - 12x + 12$

$$\ldots = \int \frac{(3x^2-12x+12)-5x+9}{(x-2)^3}dx = \ln|(x-2)^3| + \int \frac{-5(x-2)-1}{(x-2)^3}dx =$$
$$= 3\ln|x-2| - 5\int \frac{dx}{(x-2)^2} - \int \frac{dx}{(x-2)^3} = 3\ln|x-2| + \frac{5}{x-2} + \frac{1}{2(x-2)^2} + C$$

16.91

$$\int \frac{dx}{(x^2+4x+8)^3} = \int \frac{dx}{[(x+2)^2+2^2]^3} = \frac{1}{(2^2)^3}\int \frac{dx}{\left[\left(\frac{x+2}{2}\right)^2+1\right]^3} = \begin{vmatrix} t = \frac{x+2}{2} \\ dt = \frac{1}{2}dx \\ 2dt = dx \end{vmatrix} =$$
$$= \frac{1}{32}\int \frac{dt}{(t^2+1)^3} = \ldots$$

korzystając z wzoru rekurencyjnego pod zadaniem (16.89):

$$\ldots = \frac{1}{32}\left[\frac{1}{4}\cdot\frac{t}{(t^2+1)^2} + \frac{3}{4}\int\frac{dt}{(t^2+1)^2}\right] =$$
$$= \frac{1}{32}\left[\frac{t}{4(t^2+1)^2} + \frac{3}{4}\left(\frac{1}{2}\cdot\frac{t}{t^2+1} + \frac{1}{2}\int\frac{dt}{x^2+1}\right)\right] =$$
$$= \frac{1}{32}\left[\frac{t}{4(t^2+1)^2} + \frac{3t}{8(t^2+1)} + \frac{3}{8}\arctan t\right] + C =$$
$$= \frac{1}{16}\cdot\frac{x+2}{(x^2+4x+8)^2} + \frac{3}{128}\cdot\frac{x+2}{x^2+4x+8} + \frac{3}{256}\arctan\left(\frac{x+2}{2}\right) + C$$

16.92

$$\int \frac{x^3-2x^2+7x+4}{(x-1)^2(x+1)^2}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{x^3-2x^2+7x+4}{(x-1)^2(x+1)^2} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$
$$x^3-2x^2+7x+4 \equiv A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$$
$$x^3-2x^2+7x+4 \equiv (A+C)x^3 + (A+B-C+D)x^2 + (-A+2B-C-2D)x + (-A+B+C+D)$$
$$\begin{cases} A+C=1 \\ A+B-C+D=-2 \\ -A+2B-C-2D=7 \\ -A+B+C+D=4 \end{cases}$$
$$\begin{cases} A=-1 \\ B=\frac{5}{2} \\ C=2 \\ D=-\frac{3}{2} \end{cases}$$
$$\ldots = \int\frac{-dx}{x-1} + \int\frac{\frac{5}{2}dx}{(x-1)^2} + \int\frac{2dx}{x+1} + \int\frac{-\frac{3}{2}dx}{(x+1)^2} =$$

$$= -\ln|x-1| + \frac{5}{2(x-1)} + 2\ln|x+1| + \frac{3}{2(x+1)} + C$$

16.93

$$\int \frac{dx}{x^4+64} = \int \frac{dx}{(x^2-4x+8)(x^2+4x+8)} = \dots$$

rozkład na ułamki proste:

$$\frac{1}{(x^2-4x+8)(x^2+4x+8)} \equiv \frac{Ax+B}{x^2-4x+8} + \frac{Cx+D}{x^2+4x+8}$$
$$1 \equiv (Ax+B)(x^2+4x+8) + (Cx+D)(x^2-4x+8)$$
$$1 \equiv (A+C)x^3 + (4A+B-4C+D)x^2 + (8A+4B+8C-4D)x + (8B+8D)$$
$$\begin{cases} A+C=0 \\ 4A+B-4C+D=0 \\ 8A+4B+8C-4D=0 \\ 8B+8D=1 \end{cases}$$
$$\begin{cases} A=-\frac{1}{64} \\ B=\frac{1}{16} \\ C=\frac{1}{64} \\ D=\frac{1}{16} \end{cases}$$
$$\dots = \int \frac{-\frac{1}{64}x+\frac{1}{16}}{x^2-4x+8} + \int \frac{\frac{1}{64}x+\frac{1}{16}}{x^2+4x+8} =$$
$$= \int \frac{-\frac{1}{128}(x^2-4x+8)'+\frac{1}{32}}{(x-2)^2+2^2} + \int \frac{\frac{1}{128}(x^2+4x+8)'+\frac{1}{32}}{(x+2)^2+2^2} =$$
$$= -\frac{1}{128}\ln|x^2-4x+8| + \frac{1}{64}\arctan\left(\frac{x-2}{2}\right) + \frac{1}{128}\ln|x^2+4x+8| + \frac{1}{64}\arctan\left(\frac{x+2}{2}\right) + C$$

16.94

$$\int \frac{5x^3-11x^2+5x+4}{(x-1)^4}dx = \int \frac{5(x^3-3x^2+3x-1)+4x^2-10x+9}{(x-1)^4}dx =$$
$$= \int \frac{5}{x-1}dx + \int \frac{4(x^2-2x+1)-2x+5}{(x-1)^4}dx =$$
$$= 5\ln|x-1| + \int \frac{4}{(x-1)^2}dx + \int \frac{-2(x-1)+3}{(x-1)^4}dx =$$
$$= 5\ln|x-1| - \frac{4}{x-1} + \int \frac{-2}{(x-1)^3}dx + \int \frac{3}{(x-1)^4}dx =$$
$$= 5\ln|x-1| - \frac{4}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3} + C$$

16.95

$$\int \frac{dx}{x^4+6x^2+25} = \int \frac{dx}{(x^2-2x+5)(x^2+2x+5)} = \dots$$

rozkład na ułamki proste:

$$\frac{1}{(x^2-2x+5)(x^2+2x+5)} \equiv \frac{Ax+B}{x^2-2x+5} + \frac{Cx+D}{x^2+2x+5}$$
$$1 \equiv (Ax+B)(x^2+2x+5) + (Cx+D)(x^2-2x+5)$$
$$1 \equiv (A+C)x^3 + (2A+B-2C+D)x^2 + (5A+2B+5C-2D)x + (5B+5D)$$
$$\begin{cases} A+C=0 \\ 2A+B-2C+D=0 \\ 5A+2B+5C-2D=0 \\ 5B+5D=1 \end{cases}$$
$$\begin{cases} A=-\frac{1}{20} \\ B=\frac{1}{10} \\ C=\frac{1}{20} \\ D=\frac{1}{10} \end{cases}$$
$$\ldots = \int \frac{-\frac{1}{20}x+\frac{1}{10}}{x^2-2x+5} + \int \frac{\frac{1}{20}x+\frac{1}{10}}{x^2+2x+5} =$$
$$= \int \frac{-\frac{1}{40}(x^2-2x+5)'+\frac{1}{20}}{(x-1)+2^2} + \int \frac{\frac{1}{40}(x^2+2x+5)'+\frac{1}{20}}{(x+1)^2+2^2} =$$
$$= -\frac{1}{40}\ln|x^2-2x+5| + \frac{1}{40}\arctan\left(\frac{x-1}{2}\right) + \frac{1}{40}\ln|x^2+2x+5| + \frac{1}{40}\arctan\left(\frac{x+1}{2}\right) + C$$

16.96

$$\int \frac{9x^4-3x^3-23x^2+30x-1}{(x-1)^4(x+3)}dx$$

rozkład na ułamki proste:

$$\frac{9x^4-3x^3-23x^2+30x-1}{(x-1)^4(x+3)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} + \frac{E}{x+3}$$
$$9x^4-3x^3-23x^2+30x-1 \equiv A(x-1)^3(x+3) + B(x-1)^2(x+3) + C(x-1)(x+3) + \\ +D(x+3) + E(x-1)^4$$
$$9x^4-3x^3-23x^2+30x-1 \equiv (A+E)x^4 + (B-4E)x^3 + (-6A+B+C+6E)x^2 + \\ +(8A-5B+2C+D-4E)x + (-3A+3B-3C+3D+E)$$
$$\begin{cases} A+E=9 \\ B-4E=-3 \\ -6A+B+C+6E=-23 \\ 8A-5B+2C+D-4E=30 \\ -3A+3B-3C+3D+E=-1 \end{cases}$$
$$\ldots$$
$$\begin{cases} A=7 \\ B=5 \\ C=2 \\ D=3 \\ E=2 \end{cases}$$

$$\ldots = \int \frac{7}{x-1}dx + \int \frac{5}{(x-1)^2}dx + \int \frac{2}{(x-1)^3}dx + \int \frac{3}{(x-1)^4}dx + \int \frac{2}{x+3} =$$
$$= 7\ln|x-1| - \frac{5}{x-1} - \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3} + 2\ln|x+3| + C$$

16.97

$$\int \frac{x^3-2x^2+5x-8}{x^4+8x^2+16}dx = \int \frac{x^3-2x^2+5x-8}{(x^2+4)^2}dx = \int \frac{x(x^2+4)-2(x^2+4)+x}{(x^2+4)^2}dx =$$
$$= \int \frac{x}{x^2+4} - 2\int \frac{dx}{x^2+2^2} + \int \frac{x}{(x^2+4)^2} = \frac{1}{2}\ln|x^2+4| - \arctan\left(\frac{x}{2}\right) - \frac{1}{2(x^2+4)} + C$$
$$\int \frac{x}{(x^2+4)^2} = \begin{vmatrix} t = x^2+4 \\ dt = 2xdx \\ \frac{1}{2}dt = xdx \end{vmatrix} = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t} + C = -\frac{1}{2(x^2+4)} + C$$

16.98

$$\int \frac{3x^2+x-2}{(x-1)^3(x^2+1)}dx = \ldots$$

rozkład na ułamki proste:

$$\frac{3x^2+x-2}{(x-1)^3(x^2+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1}$$
$$3x^2+x-2 \equiv A(x-1)^2(x^2+1) + B(x-1)(x^2+1) + C(x^2+1) + (Dx+E)(x-1)^3$$
$$3x^2+x-2 \equiv (A+D)x^4 + (-2A+B-3D+E)x^3 + (2A-B+C+3D-3E)x^2 +$$
$$+(-2A+B-D+3E)x + (A-B+C-E)$$

$$\begin{cases} A+D=0 \\ -2A+B-3D+E=0 \\ 2A-B+C+3D-3E=3 \\ -2A+B-D+3E=1 \\ A-B+C-E=-2 \end{cases}$$

$$\begin{cases} A=-\frac{3}{2} \\ B=\frac{5}{2} \\ C=1 \\ D=\frac{3}{2} \\ E=-1 \end{cases}$$

$$\ldots = \int \frac{-\frac{3}{2}}{x-1}dx + \int \frac{\frac{5}{2}}{(x-1)^2}dx + \int \frac{1}{(x-1)^3}dx + \int \frac{\frac{3}{2}x-1}{x^2+1}dx =$$
$$= -\frac{3}{2}\ln|x-1| - \frac{5}{2(x-1)} - \frac{1}{2(x-1)^2} + \int \frac{\frac{3}{4}(x^2+1)'-1}{x^2+1}dx =$$
$$= -\frac{3}{2}\ln|x-1| - \frac{5}{2(x-1)} - \frac{1}{2(x-1)^2} + \frac{3}{4}\ln|x^2+1| - \arctan x + C$$

17 Całki funkcji niewymiernych.

17.1 § Całki funkcji zawierających pierwiastki z wyrażenia liniowego.

17.6

$$\int \sqrt{2x+1}dx = \begin{vmatrix} t = 2x+1 \\ \frac{1}{2}dt = dx \end{vmatrix} = \frac{1}{2}\int \sqrt{t}dt = \frac{1}{3}t^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$$

17.7

$$\int \frac{dx}{\sqrt{3+4x}} = \begin{vmatrix} t = \sqrt{3+4x} \\ t^2 = 3+4x \\ 2tdt = 4dx \\ \frac{1}{2}tdt = dx \end{vmatrix} = \int \frac{\frac{1}{2}t}{t}dt = \frac{1}{2}t + C = \frac{1}{2}\sqrt{3+4x} + C$$

17.8

$$\int \frac{dx}{\sqrt[3]{3x-4}} = \begin{vmatrix} t = \sqrt[3]{3x-4} \\ t^3 = 3x-4 \\ 3t^2dt = 3dx \\ t^2dt = dx \end{vmatrix} = \int \frac{t^2}{t}dt = \frac{1}{2}t^2 + C = \frac{1}{2}(3x-4)^{\frac{2}{3}} + C$$

17.9

$$\int \frac{dx}{\sqrt[5]{(2x+1)^3}} = \begin{vmatrix} t = \sqrt[5]{2x+1} \\ t^5 = 2x+1 \\ 5t^4dt = 2dx \\ \frac{5}{2}t^4dt = dx \end{vmatrix} = \int \frac{\frac{5}{2}t^4dt}{t^3} = \frac{5}{4}t^2 + C = \frac{5}{4}\sqrt[5]{(2x+1)^2} + C$$

17.10

$$\int x\sqrt[3]{x-4}dx = \begin{vmatrix} t = \sqrt[3]{x-4} \\ t^3 = x-4 \\ 3t^2dt = dx \\ x = t^3+4 \end{vmatrix} = \int (t^3+4)t \cdot 3t^2dt = \int (3t^6+12t^3)dt =$$

$$= \frac{3}{7}t^7 + 3t^4 + C = \frac{3}{7}t^4(t^3+7) + C = \frac{3}{7}\sqrt[3]{x-4}(x-4)(x-4+7) + C =$$

$$= \frac{3}{7}(x-4)(x+3)\sqrt[3]{x-4} + C = \frac{3}{7}(x^2-x-12)\sqrt[3]{x-4} + C$$

17.11

$$\int x\sqrt[3]{3x-1}dx = \begin{vmatrix} t = \sqrt[3]{3x-1} \\ t^3 = 3x-1 \\ 3t^2dt = 3dx \\ t^2dt = dx \\ x = \frac{t^3+1}{3} \end{vmatrix} = \int \frac{t^3+1}{3} \cdot t \cdot t^2dt = \frac{1}{3}\int (t^6+t^3)dt =$$

$$= \frac{1}{21}t^7 + \frac{1}{12}t^4 + C = \frac{1}{21}(3x-1)^{\frac{7}{3}} + \frac{1}{12}(3x-1)^{\frac{4}{3}} + C$$

17.12

$$\int x\sqrt{2+3x}dx = \begin{vmatrix} t = \sqrt{2+3x} \\ t^2 = 2+3x \\ 2tdt = 3dx \\ \frac{2}{3}tdt = dx \\ x = \frac{t^2-2}{3} \end{vmatrix} = \int \frac{t^2-2}{3} \cdot t \cdot \frac{2}{3}tdt = \frac{2}{9}\int (t^4 - 2t^2)dt =$$

$$= \frac{2}{45}t^5 - \frac{4}{27}t^3 + C = \frac{2}{45}(2+3x)^{\frac{5}{2}} - \frac{4}{27}(2+3x)^{\frac{3}{2}} + C$$

17.13

$$\int x\sqrt{1-5x}dx = \begin{vmatrix} t = \sqrt{1-5x} \\ t^2 = 1-5x \\ 2tdt = -5dx \\ -\frac{2}{5}tdt = dx \\ x = \frac{t^2-1}{-5} \end{vmatrix} = \int \frac{t^2-1}{-5} \cdot t \cdot \left(-\frac{2}{5}t\right) dt = \frac{2}{25}\int (t^4 - t^2)dt =$$

$$= \frac{2}{125}t^5 - \frac{2}{75}t^3 + C = \frac{2}{125}(1-5x)^{\frac{5}{2}} - \frac{2}{75}(1-5x)^{\frac{3}{2}} + C$$

17.14

$\rightarrow$ **(17.10)**

17.15

$$\int \frac{xdx}{\sqrt[4]{2x+3}} = \begin{vmatrix} u = x & dv = (2x+3)^{-\frac{1}{4}}dx \\ du = dx & v = \frac{2}{3}(2x+3)^{\frac{3}{4}} \end{vmatrix} = \frac{2}{3}x(2x+3)^{\frac{3}{4}} - \int \frac{2}{3}(2x+3)^{\frac{3}{4}}dx =$$
$$= \frac{2}{3}x(2x+3)^{\frac{3}{4}} - \frac{4}{21}(2x+3)^{\frac{7}{4}} + C$$

17.16

$$\int \frac{x^2dx}{3\sqrt[3]{x+2}} = \begin{vmatrix} t = \sqrt[3]{x+2} \\ t^3 = x+2 \\ x = t^3-2 \\ dx = 3t^2dt \\ x^2 = (t^3-2)^2 \end{vmatrix} = \int (t^3-2)^2 tdt = \int (t^7 - 4t^4 + 4t)dt =$$

$$= \frac{1}{8}t^8 - \frac{4}{5}t^5 + 2t^2 + C = \frac{1}{8}(x+2)^{\frac{8}{3}} - \frac{4}{5}(x+2)^{\frac{5}{3}} + 2(x+2)^{\frac{2}{3}} + C$$

17.17

$$\int \frac{x^2+1}{\sqrt{3x+1}}dx = \int \frac{x^2 dx}{\sqrt{3x+1}} + \int \frac{dx}{\sqrt{3x+1}} = \left| \begin{array}{l} t = \sqrt{3x+1} \\ t^2 = 3x+1 \\ x = \frac{t^2-1}{3} \\ dx = \frac{2}{3}tdt \end{array} \right| =$$

$$= \int \frac{2}{3}\left(\frac{t^2-1}{3}\right)^2 dt + \frac{2}{3}t = \frac{2}{27}\int (t^4-2t^2+1)dt + \frac{2}{3}t = \frac{2}{135}t^5 - \frac{4}{81}t^3 + \frac{20}{27}t + C =$$
$$= \frac{2}{135}(3x+1)^{\frac{5}{2}} - \frac{4}{81}(3x+1)^{\frac{3}{2}} + \frac{20}{27}\sqrt{3x+1} + C$$

17.18

$$\int x\sqrt[4]{2x+3}dx = \left| \begin{array}{ll} u = x & dv = \sqrt[4]{2x+3}dx \\ du = dx & v = \frac{2}{5}(2x+3)^{\frac{5}{4}} \end{array} \right| = \frac{2}{5}x(2x+3)^{\frac{5}{4}} - \int \frac{2}{5}(2x+3)^{\frac{5}{4}}dx =$$
$$= \frac{2}{5}x(2x+3)^{\frac{5}{4}} - \frac{4}{45}(2x+3)^{\frac{9}{4}} + C$$

17.19

$$\int \frac{dx}{x\sqrt{x+a}} = \left| \begin{array}{c} t = \sqrt{x+a} \\ t^2 = x+a \\ x = t^2-a \\ dx = 2tdt \end{array} \right| = \int \frac{2}{t^2-a}dt = -2\int \frac{dt}{(\sqrt{a})^2-t^2} = -\frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a}+t}{\sqrt{a}-t}\right| + C =$$

$$= \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a}-\sqrt{x+a}}{\sqrt{a}+\sqrt{x+a}}\right| + C = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{x+a}-\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right| + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \text{ gdzie } a > 0 \wedge |x| \neq a$$

17.20

$$\int \frac{dx}{x\sqrt{x-a}} = \left| \begin{array}{c} t = \sqrt{x-a} \\ t^2 = x-a \\ x = t^2+a \\ dx = 2tdt \end{array} \right| = \int \frac{2}{t^2+a}dt = \frac{2}{\sqrt{a}}\arctan\left(\frac{t}{\sqrt{a}}\right) + C = \frac{2}{\sqrt{a}}\arctan\sqrt{\frac{x-a}{a}} + C$$

17.21

$$\int \frac{\sqrt{x}}{x-1} = \left| \begin{array}{c} t = \sqrt{x} \\ t^2 = x \\ dx = 2tdt \end{array} \right| = \int \frac{2t^2}{t^2-1}dt = \int 2dt + \int \frac{2}{t^2-1}dt = 2t - 2\int \frac{dt}{1-t^2} =$$

$$= 2t - \ln\left|\frac{1+t}{1-t}\right| + C = 2\sqrt{x} - \ln\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right| + C = 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$

17.22

$$\int \frac{\sqrt{x+1}}{x}dx = \left|\begin{array}{c} t = \sqrt{x+1} \\ t^2 = x+1 \\ x = t^2-1 \\ dx = 2tdt \end{array}\right| = \int \frac{2t^2}{t^2-1} = ...$$

korzystając z całki obliczonej w przykładzie (17.21) ostatecznie otrzymujemy:

$$... = 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

17.23

$$\int \frac{1+\sqrt{x}}{1-\sqrt{x}}dx = \left|\begin{array}{c} t = \sqrt{x} \\ t^2 = x \\ 2tdt = dx \end{array}\right| = \int \frac{(1+t)\cdot 2t}{1-t}dt = -2\int \frac{t^2+t}{t-1}dt = -2\int \frac{t(t-1)+2t}{t-1}dt =$$

$$= -2\int tdt - 2\int \frac{2(t-1)+2}{t-1} = -t^2 - 2\int 2dt - 4\int \frac{dt}{t-1} =$$

$$= -t^2 - 4t - 4\ln|t-1| + C = -x - 4\sqrt{x} - 4\ln|\sqrt{x}-1| + C$$

17.24

$$\int \frac{dx}{(x+1)\sqrt{1-x}} = \left|\begin{array}{c} t = \sqrt{1-x} \\ t^2 = 1-x \\ -t^2+1 = x \\ -2tdt = dx \end{array}\right| = \int \frac{-2t}{(-t^2+2)t}dt = \int \frac{2dt}{t^2-2} =$$

$$= -2\int \frac{dt}{(\sqrt{2})^2 - t^2} = -\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + C = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{1-x}-\sqrt{2}}{\sqrt{1-x}+\sqrt{2}}\right| + C$$

17.25

$$\int \sqrt{1+\sqrt{x}}dx = \left|\begin{array}{c} t = \sqrt{x} \\ t^2 = x \\ 2tdt = dx \end{array}\right| = 2\int t\sqrt{t+1}dt = \left|\begin{array}{cc} u = 2t & dv = \sqrt{t+1}dt \\ du = 2dt & v = \frac{2}{3}(t+1)^{\frac{3}{2}} \end{array}\right| =$$

$$= \frac{4}{3}t(t+1)^{\frac{3}{2}} - \frac{4}{3}\int (t+1)^{\frac{3}{2}}dt = \frac{4}{3}t(t+1)^{\frac{3}{2}} - \frac{8}{15}(t+1)^{\frac{5}{2}} + C =$$

$$= \frac{4}{3}\sqrt{x}(\sqrt{x}+1)^{\frac{3}{2}} - \frac{8}{15}(\sqrt{x}+1)^{\frac{5}{2}} + C$$

17.26

$$\int \frac{\sqrt[3]{x}dx}{x+\sqrt[6]{x^5}} = \left| \begin{matrix} t=\sqrt[6]{x} \\ t^6=x \\ 6t^5=dx \end{matrix} \right| = \int \frac{6t^7}{t^6+t^5}dt = 6\int \frac{t^2}{t+1}dt = 6\int \frac{(t-1)(t+1)+1}{t+1}dt =$$

$$= 6\int (t-1)dt + 6\int \frac{dt}{t+1} = 3t^2-6t+6ln|t+1|+C = 3\sqrt[3]{x}-6\sqrt[6]{x}+6\ln|\sqrt[6]{x}+1|+C$$

17.27

$$\int \frac{dx}{\sqrt{x}+2\sqrt[3]{x^2}} = \left| \begin{matrix} t=\sqrt[6]{x} \\ t^6=x \\ 6t^5=dx \end{matrix} \right| = \int \frac{6t^5dt}{t^3+2t^4} = \int \frac{6t^2}{2t+1}dt = \int \frac{3t(2t+1)-3t}{2t+1}dt =$$

$$= \int 3tdt + \int \frac{-\frac{3}{2}(2t+1)+\frac{3}{2}}{2t+1}dt = \frac{3}{2}t^2 - \int \frac{3}{2}dt + \frac{3}{2}\int \frac{dt}{2t+1} =$$

$$= \frac{3}{2}t^2 - \frac{3}{2}t + \frac{3}{4}\ln|2t+1| + C = \frac{3}{2}\sqrt[3]{x} - \frac{3}{2}\sqrt[6]{x} + \frac{3}{4}\ln|2\sqrt[6]{x}+1| + C$$

17.28

$$\int \frac{dx}{\sqrt{x-5}+\sqrt{x-7}} = \frac{1}{2}\int (\sqrt{x-5}-\sqrt{x-7})dx = \frac{1}{3}\left[(x-5)^{\frac{3}{2}}-(x-7)^{\frac{3}{2}}\right]+C$$

17.29

$$\int \frac{dx}{x\sqrt{x+9}} = \left| \begin{matrix} t=\sqrt{x+9} \\ t^2=x+9 \\ t^2-9=x \\ 2tdt=dx \end{matrix} \right| = \int \frac{2dt}{t^2-9} = -2\int \frac{dt}{3^2-t^2} = -\frac{1}{3}\ln\left|\frac{3+t}{3-t}\right| + C =$$

$$= \frac{1}{3}\ln\left|\frac{t-3}{t+3}\right| + C = \frac{1}{3}\ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right| + C$$

17.30

$$\int x^2\sqrt[3]{7-2x}dx = \left| \begin{matrix} u=x^2 & dv=\sqrt[3]{7-2x}dx \\ du=2xdx & v=-\frac{3}{8}(7-2x)^{\frac{4}{3}} \end{matrix} \right| =$$

$$= -\frac{3}{8}x^2(7-2x)^{\frac{4}{3}} + \frac{3}{4}\int x(7-2x)^{\frac{4}{3}}dx = \left| \begin{matrix} u=x & dv=(7-2x)^{\frac{4}{3}}dx \\ du=dx & v=-\frac{3}{14}(7-2x)^{\frac{7}{3}} \end{matrix} \right| =$$

$$= -\frac{3}{8}x^2(7-2x)^{\frac{4}{3}} - \frac{9}{56}x(7-2x)^{\frac{7}{3}} + \frac{9}{56}\int (7-2x)^{\frac{7}{3}}dx =$$

$$= -\frac{3}{8}x^2(7-2x)^{\frac{4}{3}} - \frac{9}{56}x(7-2x)^{\frac{7}{3}} - \frac{27}{1120}(7-2x)^{\frac{10}{3}} + C$$

17.31

$$\int \frac{dx}{\sqrt{x+1}+\sqrt[3]{x+1}} = \left| \begin{array}{c} t=\sqrt[6]{x+1} \\ t^2=\sqrt[3]{x+1} \\ t^3=\sqrt{x+1} \\ t^6=x+1 \\ 6t^5dt=dx \end{array} \right| = \int \frac{6t^5dt}{t^3+t^2} = 6\int \frac{t^3dt}{t+1} = 6\int \frac{(t^2-t+1)(t+1)-1}{t+1}dt =$$

$$= 6\int (t^2-t+1)dt - 6\int \frac{dt}{t+1} = 2t^3-3t^2+6t-6\ln|t+1|+C =$$

$$= 2\sqrt{x+1} - 3\sqrt[3]{x+1} + 6\sqrt[6]{x+1} - 6\ln|\sqrt[6]{x+1}+1| + C$$

17.32

$$\int \sqrt{\frac{x-1}{x-2}} \cdot \frac{dx}{(x-1)^2} = \left| \begin{array}{c} t=\sqrt{\frac{x-1}{x-2}} \\ t^2=\frac{x-1}{x-2} \\ \frac{1}{t^2}=\frac{x-2}{x-1} \\ \frac{1}{t^2}-1=-\frac{1}{x-1} \\ \frac{-2dt}{t^3}=\frac{dx}{(x-1)^2} \end{array} \right| = \int \frac{-2tdt}{t^3} = \int \frac{-2dt}{t^2} = \frac{2}{t} + C = 2\sqrt{\frac{x-2}{x-1}} + C$$

17.33

$$\int \sqrt{\frac{1-x}{1+x}} \cdot \frac{dx}{x} = \left| \begin{array}{cc} t=\sqrt{\frac{1-x}{1+x}} & t^2=\frac{1-x}{1+x} \\ -t^2=\frac{x-1}{x+1} & -t^2-1=-\frac{2}{x+1} \\ \frac{2}{t^2+1}=x+1 & \frac{-4tdt}{(t^2+1)^2}=dx \\ \frac{-t^2+1}{t^2+1}=x & \frac{t^2+1}{-t^2+1}=\frac{1}{x} \end{array} \right| = \int t \cdot \frac{-4tdt}{(t^2+1)^2} \cdot \frac{t^2+1}{-t^2+1} =$$

$$= \int \frac{4t^2}{(t^2+1)(t-1)(t+1)}dt = \ldots$$

rozkład na ułamki proste:

$$\frac{4t^2}{(t^2+1)(t-1)(t+1)} \equiv \frac{At+B}{t^2+1} + \frac{C}{t-1} + \frac{D}{t+1}$$
$$4t^2 \equiv (At+B)(t^2-1) + C(t^3+t^2+t+1) + D(t^3-t^2+t-1)$$
$$4t^2 \equiv (A+C+D)t^3 + (B+C-D)t^2 + (-A+C+D)t + (-B+C-D)$$
$$\begin{cases} A+C+D=0 \\ B+C-D=4 \\ -A+C+D=0 \\ -B+C-D=0 \end{cases}$$

$$\begin{cases} A = 0 \\ B = 2 \\ C = 1 \\ D = -1 \end{cases}$$

$$... = \int \frac{2dt}{t^2+1} + \int \frac{dt}{t-1} - \int \frac{dt}{t+1} = 2\arctan t + \ln|t-1| - \ln|t+1| + C =$$

$$= 2\arctan\left(\sqrt{\frac{1-x}{1+x}}\right) + \ln\left|\sqrt{\frac{1-x}{1+x}} - 1\right| - \ln\left|\sqrt{\frac{1-x}{1+x}} + 1\right| + C$$

17.34

$$\int \frac{xdx}{\sqrt[3]{x+1} - \sqrt{x+1}} = \begin{vmatrix} t = \sqrt[6]{x+1} \\ t^6 = x+1 \\ t^6 - 1 = x \\ 6t^5 dt = dx \end{vmatrix} = \int \frac{(t^6-1)\cdot 6t^5 dt}{t^2 - t^3} = -6\int \frac{t^3(t^6-1)}{t-1}dt =$$

$$= -6\int t^3(t^5+t^4+t^3+t^2+t+1)dt = -\frac{2}{3}t^9 - \frac{3}{4}t^8 - \frac{6}{7}t^7 - t^6 - \frac{6}{5}t^5 - \frac{3}{2}t^4 + C =$$

$$= -\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{3}{4}(x+1)^{\frac{4}{3}} - \frac{6}{7}(x+1)^{\frac{7}{6}} - (x+1) - \frac{6}{5}(x+1)^{\frac{5}{6}} - \frac{3}{2}(x+1)^{\frac{2}{3}} + C$$

17.35

$$\int \frac{\sqrt[3]{x^2} - \sqrt{x} + 1}{\sqrt[3]{x} - 1} = \begin{vmatrix} t = \sqrt[6]{x} \\ t^6 = x \end{vmatrix} = \int \frac{(t^4 - t^3 + 1)\cdot 6t^5 dt}{t^2-1} = 6\int \frac{t^9 - t^8 + t^5}{t^2-1}dt = ...$$

pisemne dzielenie wielomianów:

$$\begin{array}{l}
(t^9 - t^8 + t^5) \quad : \quad (t^2 - 1) = t^7 - t^6 + t^5 - t^4 + 2t^3 - t^2 + 2t - 1 \\
\underline{-t^9 + t^7} \\
\quad -t^8 + t^7 + t^5 \\
\quad \underline{t^8 - t^6} \\
\qquad t^7 - t^6 + t^5 \\
\qquad \underline{-t^7 + t^5} \\
\qquad\quad -t^6 + 2t^5 \\
\qquad\quad \underline{t^6 - t^4} \\
\qquad\quad 2t^5 - t^4 \\
\qquad\quad \underline{-2t^5 + 2t^3} \\
\qquad\quad -t^4 + 2t^3 \\
\qquad\quad \underline{t^4 - t^2} \\
\qquad\quad 2t^3 - t^2 \\
\qquad\quad \underline{-2t^3 + 2t} \\
\qquad\qquad -t^2 + 2t \\
\qquad\qquad \underline{\underline{t^2 - 1}} \\
\qquad\qquad 2t - 1
\end{array}$$

$$... = 6\int\left(t^7 - t^6 + t^5 - t^4 + 2t^3 - t^2 + 2t - 1 + \frac{2t-1}{t^2-1}\right)dt =$$

$$= \frac{3}{4}t^8 - \frac{6}{7}t^7 + t^6 - \frac{6}{5}t^5 + 3t^4 - 2t^3 + 6t^2 - 6t + 6\ln|t^2+1| + 6\int\frac{dt}{1-t^2} =$$

$$= \frac{3}{4}x^{\frac{4}{3}} - \frac{6}{7}x^{\frac{7}{6}} + x - \frac{6}{5}x^{\frac{5}{6}} + 3x^{\frac{2}{3}} - 2\sqrt{x} + 6\sqrt[3]{x} - 6\sqrt[6]{x} + 6\ln|\sqrt[3]{x}+1| + 3\ln\left|\frac{1+\sqrt[6]{x}}{1-\sqrt[6]{x}}\right| + C$$

17.2 § Całki funkcji zawierających pierwiastek kwadratowy z trójmianu kwadratowego

17.51

$$\int\frac{(8x+3)dx}{\sqrt{4x^2+3x+1}} = 2\int\frac{(4x^2+3x+1)'}{2\sqrt{4x^2+3x+1}}dx = 2\sqrt{4x^2+3x+1} + C$$

$$\int\frac{f'(x)}{2\sqrt{f(x)}}dx = \sqrt{f(x)} + C$$

17.52

$$\int\frac{(10x+15)dx}{\sqrt{36x^2+108x+77}} = \int\frac{\frac{5}{18}(36x^2+108x+77)'}{2\sqrt{36x^2+108x+77}}dx = \frac{5}{18}\sqrt{36x^2+108x+77} + C$$

17.53

$$\int\frac{dx}{\sqrt{2x-x^2}} = \int\frac{dx}{\sqrt{1-(x-1)^2}} = \arcsin(x-1) + C$$

17.54

$$\int\frac{dx}{\sqrt{7-6x-x^2}} = \int\frac{dx}{\sqrt{4^2-(x+3)^2}} = \arcsin\left(\frac{x+3}{4}\right) + C$$

17.55

$$\int\frac{dx}{\sqrt{1-9x^2}} = \left|\begin{matrix} t = 3x \\ \frac{1}{3}dt = dx\end{matrix}\right| = \frac{1}{3}\int\frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\arcsin(t) + C = \frac{1}{3}\arcsin(3x) + C$$

17.56

$$\int\frac{dx}{\sqrt{(2r-x)x}} = \int\frac{dx}{\sqrt{r^2-(x-r)^2}} = \arcsin\left(\frac{x-r}{r}\right) + C$$

17.57

$$\int \frac{(x+3)dx}{\sqrt{1-4x^2}} = \int \frac{-\frac{1}{4}(1-4x^2)'}{2\sqrt{1-4x^2}}dx + \int \frac{3dx}{\sqrt{1-(2x)^2}} = -\frac{1}{4}\sqrt{1-4x^2} + \frac{3}{2}\arcsin(2x) + C$$

17.58

$$\int \frac{xdx}{\sqrt{1-2x-3x^2}} = \int \frac{-\frac{1}{3}(1-2x-3x^2)'}{2\sqrt{1-2x-3x^2}}dx - \int \frac{\frac{1}{3}dx}{\sqrt{1-2x-3x^2}} =$$
$$= -\frac{1}{3}\sqrt{1-2x-3x^2} - \frac{1}{3\sqrt{3}}\int \frac{dx}{\sqrt{\frac{1}{3}-\frac{2}{3}x+x^2}} = -\frac{1}{3}\sqrt{1-2x-3x^2} - \frac{1}{3\sqrt{3}}\int \frac{dx}{\sqrt{(\frac{2}{3})^2-(x+\frac{1}{3})^2}} =$$

$$= -\frac{1}{3}\sqrt{1-2x-3x^2} - \frac{1}{3\sqrt{3}}\arcsin\left(\frac{3x+1}{2}\right) + C$$

17.59

$$\int \sqrt{1-4x^2}dx = \begin{vmatrix} t = 2x \\ dt = 2dx \\ \frac{1}{2}dt = dx \end{vmatrix} = \frac{1}{2}\int \sqrt{1-t^2}dt = \frac{1}{2}\left(\frac{1}{2}\arcsin(t) - \frac{1}{2}t\sqrt{1-t^2}\right) + C =$$
$$= \frac{1}{4}\arcsin(2x) - \frac{1}{2}x\sqrt{1-4x^2} + C$$

$$\boxed{\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\arcsin\frac{x}{|a|} + \frac{x}{2}\sqrt{a^2-x^2} + C}$$

17.60

$$\int \frac{6x+5}{\sqrt{6+x-x^2}}dx = \int \frac{-3(6+x-x^2)'}{\sqrt{6+x-x^2}}dx + \int \frac{8dx}{\sqrt{6+x-x^2}} =$$
$$= \int \frac{-6(6+x-x^2)'}{2\sqrt{6+x-x^2}}dx + \int \frac{8dx}{\sqrt{(\frac{5}{2})^2-(x-\frac{1}{2})^2}} = -6\sqrt{6+x-x^2} + 8\arcsin\left(\frac{2x-1}{5}\right) + C$$

17.61

$$\int \frac{x-5}{\sqrt{5+4x-x^2}}dx = \int \frac{-\frac{1}{2}(5+4x-x^2)'}{\sqrt{5+4x-x^2}}dx - \int \frac{3dx}{\sqrt{5+4x-x^2}} =$$
$$= -\sqrt{5+4x-x^2} - 3\int \frac{dx}{\sqrt{3^2-(x-2)^2}} = -\sqrt{5+4x-x^2} - 3\arcsin\left(\frac{x+2}{3}\right) + C$$

17.62

$$\int \frac{x+1}{\sqrt{8+2x-x^2}}dx = \int \frac{-\frac{1}{2}(8+2x-x^2)}{\sqrt{8+2x-x^2}}dx + \int \frac{2dx}{\sqrt{8+2x-x^2}} =$$

$$= -\sqrt{8+2x-x^2} + \int \frac{2dx}{\sqrt{3^2-(x-1)^2}} = -\sqrt{8+2x-x^2} + 2\arcsin\left(\frac{x-1}{3}\right) + C$$

17.63

$$\int \sqrt{6x-x^2}dx = \int \sqrt{3^2-(x-3)^2}dx = \frac{9}{2}\arcsin\left(\frac{x-3}{3}\right) + \frac{1}{2}(x-3)\sqrt{6x-x^2} + C$$

17.64

$$\int \frac{2x-3}{\sqrt{3-2x-x^2}}dx = \int \frac{-(3-2x-x^2)'}{\sqrt{3-2x-x^2}}dx - \int \frac{5dx}{\sqrt{3-2x-x^2}} =$$
$$= -2\sqrt{3-2x-x^2} - \int \frac{5dx}{\sqrt{2^2-(x+1)^2}} = -2\sqrt{3-2x-x^2} - 5\arcsin\left(\frac{x+1}{2}\right) + C$$

17.65

$$\int \frac{dx}{\sqrt{x^2+3x+2}} = \ln|x+\frac{3}{2}+\sqrt{x^2+3x+2}| + C$$

$$\int \frac{dx}{\sqrt{x^2+px+q}} = \ln|x+\tfrac{1}{2}p+\sqrt{x^2+px+q}| + C$$

17.66

$$\int \frac{dx}{\sqrt{4x^2+3x-1}} = \left|\begin{array}{c} t=2x \\ dt=2dx \\ \frac{1}{2}dt=dx \end{array}\right| = \frac{1}{2}\int \frac{dt}{\sqrt{t^2+\frac{3}{2}t-1}} = \ln|t+\frac{3}{4}+\sqrt{t^2+\tfrac{3}{2}t-1}| + C =$$
$$= \ln|2x+\frac{3}{4}+\sqrt{4x^2+3x-1}| + C$$

17.67

$$\int \frac{dx}{\sqrt{x^2-x+m}} = \ln|x-\frac{1}{2}+\sqrt{x^2-x+m}| + C$$

17.68

$$\int \frac{dx}{\sqrt{(x-a)(x-3a)}} = \int \frac{dx}{\sqrt{x^2-4ax+3a^2}} = \ln|x-2a+\sqrt{(x-a)(x-3a)}| + C$$

17.69

$$\int \frac{(x+3)dx}{\sqrt{x^2+2x}} = \int \frac{\frac{1}{2}(x^2+2x)'}{\sqrt{x^2+2x}} + \int \frac{2dx}{\sqrt{x^2+2x}} = \sqrt{x^2+2x} + 2\ln|x+1+\sqrt{x^2+2x}| + C$$

17.70

$$\int \frac{(3x+2)dx}{\sqrt{x^2-5x+19}} = \int \frac{\frac{3}{2}(x^2-5x+19)'}{\sqrt{x^2-5x+19}}dx + \int \frac{\frac{19}{2}dx}{\sqrt{x^2-5x+19}} =$$

$$= 3\sqrt{x^2-5x+19} + \frac{19}{2}\ln|x-\frac{5}{2}+\sqrt{x^2-5x+19}| + C$$

17.71

$$\int \frac{x+a}{\sqrt{x^2-ax}}dx = \int \frac{\frac{1}{2}(x^2-ax)'}{\sqrt{x^2-ax}}dx + \int \frac{\frac{3}{2}a}{\sqrt{x^2-ax}}dx =$$

$$= \sqrt{x^2-ax} + \frac{3}{2}a\ln|x-\frac{a}{2}+\sqrt{x^2-ax}| + C$$

17.72

$$\int \frac{3x-2}{\sqrt{4x^2-4x+5}}dx = \int \frac{\frac{3}{8}(4x^2-4x+5)'}{\sqrt{4x^2-4x+5}}dx - \int \frac{\frac{1}{2}dx}{\sqrt{4x^2-4+5}} =$$
$$= \frac{3}{4}\sqrt{4x^2-4x+5} - \frac{1}{4}\int \frac{dx}{\sqrt{x^2-x+\frac{5}{4}}} = \frac{3}{4}\sqrt{4x^2-4x+5} - \frac{1}{4}\ln|x-\frac{1}{2}+\sqrt{x^2-x+\frac{5}{4}}| + C$$

17.73

$$\int \frac{3x+2}{\sqrt{x^2-4x+5}}dx = \int \frac{\frac{3}{2}(x^2-4x+5)'}{\sqrt{x^2-4x+5}} + \int \frac{8dx}{\sqrt{x^2-4x+5}} =$$

$$= 3\sqrt{x^2-4x+5} + 8\ln|x-2+\sqrt{x^2-4x+5}| + C$$

17.74

$$\int \frac{3x-4}{\sqrt{4x^2+5x-8}}dx = \int \frac{\frac{3}{2}x-2}{\sqrt{x^2+\frac{5}{4}x-2}}dx = \int \frac{\frac{3}{4}(x^2+\frac{5}{4}x-2)'}{\sqrt{x^2+\frac{5}{4}x-2}}dx - \int \frac{\frac{47}{16}dx}{\sqrt{x^2+\frac{5}{4}x-2}} =$$

$$= \frac{3}{2}\sqrt{x^2+\frac{5}{4}x-2} - \frac{47}{16}\ln|x+\frac{5}{8}+\sqrt{x^2+\frac{5}{4}x-2}| + C$$

17.75

$$\int \frac{5x+2}{\sqrt{2x^2+8x-1}}dx = \int \frac{\frac{5}{4}(2x^2+8x-1)'}{\sqrt{2x^2+8x-1}} - \int \frac{8dx}{\sqrt{2x^2+8x-1}} =$$
$$= \frac{5}{2}\sqrt{2x^2+8x-1} - \frac{8}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+4x-\frac{1}{2}}} =$$

$$= \frac{5}{2}\sqrt{2x^2+8x-1} - 4\sqrt{2}\ln\left|x+2+\sqrt{x^2+4x-\frac{1}{2}}\right| + C$$

17.76

$$\int\sqrt{2x+x^2}dx = \int\sqrt{(x+1)^2-1}dx = \frac{1}{2}(x+1)\sqrt{x^2+2x} - \frac{1}{2}\ln|x+1+\sqrt{x^2+2x}| + C$$

$$\int\sqrt{x^2+k}dx = \frac{1}{2}x\sqrt{x^2+k} + \frac{1}{2}k\ln|x+\sqrt{x^2+k}| + C, \text{ gdzie } x^2+k>0$$

17.77

$$\int\frac{5x-4}{\sqrt{3x^2-2x+1}}dx = \int\frac{\frac{5}{6}(3x^2-2x+1)'}{\sqrt{3x^2-2x+1}} - \int\frac{\frac{7}{3}dx}{\sqrt{3x^2-2x+1}} =$$
$$= \frac{5}{3}\sqrt{3x^2-2x+1} - \frac{7}{3\sqrt{3}}\int\frac{dx}{\sqrt{x^2-\frac{2}{3}x+\frac{1}{3}}} =$$
$$= \frac{5}{3}\sqrt{3x^2-2x+1} - \frac{7}{3\sqrt{3}}\ln\left|x-\frac{1}{3}+\sqrt{x^2-\frac{2}{3}x+\frac{1}{3}}\right| + C$$

17.78

$$\int\sqrt{3-2x-x^2}dx = \int\sqrt{2^2-(x+1)^2}dx = 2\arcsin\left(\frac{x+1}{2}\right) + \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + C$$

17.79

$$\int\sqrt{x^2-4}dx = \frac{1}{2}x\sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + C$$

17.80

$$\int\sqrt{3x^2+10x+9}dx = \sqrt{3}\int\sqrt{x^2+\tfrac{10}{3}x+3}dx = \sqrt{3}\int\sqrt{(x+\tfrac{5}{3})^2+\tfrac{2}{9}} =$$

$$= \frac{\sqrt{3}}{2}(x+\frac{5}{3})\sqrt{x^2+\tfrac{10}{3}x+3} + \frac{\sqrt{3}}{9}\ln|x+\frac{5}{3}+\sqrt{x^2+\tfrac{10}{3}x+3}| + C$$

17.81

$$\int\sqrt{x^2-3x+2}dx = \int\sqrt{(x-\tfrac{3}{2})^2-\frac{1}{4}}dx =$$

$$= \frac{1}{2}(x-\frac{3}{2})\sqrt{x^2-3x+2} - \frac{1}{8}\ln|x-\frac{3}{2}+\sqrt{x^2-3x+2}| + C$$

17.82

$$\int \frac{x^2dx}{\sqrt{1-x^2}} = \int \frac{x^2-1}{\sqrt{1-x^2}}dx + \int \frac{dx}{\sqrt{1-x^2}} = -\int \frac{1-x^2}{\sqrt{1-x^2}}dx + \arcsin(x) =$$

$$= -\int \sqrt{1-x^2}dx + \arcsin(x) = -\frac{1}{2}\arcsin(x) - \frac{1}{2}x\sqrt{1-x^2} + \arcsin(x) + C =$$

$$= -\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$$

17.83

$$\int \frac{x^2dx}{\sqrt{x^2+2x+2}} = \int \frac{x^2+2x+2}{\sqrt{x^2+2x+2}}dx - \int \frac{2x+2}{\sqrt{x^2+2x+2}}dx =$$

$$= \int \sqrt{x^2+2x+2}dx - \int \frac{(x^2+2x+2)'}{\sqrt{x^2+2x+2}}dx = \int \sqrt{(x+1)^2+1}dx - 2\sqrt{x^2+2x+2} =$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+2} + \frac{1}{2}\ln|x+1+\sqrt{x^2+2x+2}| - 2\sqrt{x^2+2x+2} + C =$$

$$= \frac{1}{2}(x-3)\sqrt{x^2+2x+2} + \frac{1}{2}\ln|x+1+\sqrt{x^2+2x+2}| + C$$

17.84

$$\int \sqrt{\frac{x}{1-x}}dx = \left| \begin{array}{rlrl} t &= \sqrt{\frac{x}{1-x}} & t^2 &= \frac{x}{1-x} \\ t^2+1 &= \frac{1}{1-x} & 2tdt &= \frac{dx}{(x-1)^2} \\ (t^2+1)^2 &= \frac{1}{(x-1)^2} & & \end{array} \right| = \int \frac{2t^2}{(t^2+1)^2}dt = \int \frac{2(t^2+1)-2}{(t^2+1)^2}dt =$$

$$= 2\int \frac{dt}{t^2+1} - 2\int \frac{dt}{(t^2+1)^2} = 2\arctan(t) - \arctan(t) - \frac{t}{t^2+1} + C =$$

$$= \arctan\sqrt{\frac{x}{1-x}} - \sqrt{x-x^2} + C$$

17.85

$$\int \frac{2ax^2+1}{\sqrt{ax^2+2x+1}}dx, a > 1$$

metoda współczynników nieoznaczonych

$$\mathcal{I} = \int \frac{2ax^2+a}{\sqrt{ax^2+2x+1}} \equiv (Px+Q)\sqrt{ax^2+2x+1} + K\int \frac{dx}{\sqrt{ax^2+2x+1}}$$

$$\frac{2ax^2+1}{\sqrt{ax^2+2x+1}} \equiv P\sqrt{ax^2+2x+1} + \frac{(Px+Q)(ax+1)}{\sqrt{ax^2+2x+1}} + \frac{K}{\sqrt{ax^2+2x+1}}$$

$2ax^2+1 \equiv p(ax^2+2x+1)+(Px+Q)(ax+1)+K$

$$\begin{cases} 2Pa=2a \\ 3P+Qa=0 \\ P+Q+K=1 \end{cases}$$

$$\begin{cases} P=1 \\ Q=-\frac{3}{a} \\ K=\frac{3}{a} \end{cases}$$

$$\mathcal{I}=(x-\frac{3}{a})\sqrt{ax^2+2x+1}+\frac{3}{a}\int\frac{dx}{\sqrt{ax^2+2x+1}}=$$

$$=(x-\frac{3}{a})\sqrt{ax^2+2x+1}+\frac{3}{a\sqrt{a}}\int\frac{dx}{\sqrt{x^2+\frac{2}{a}x+\frac{1}{a}}}=$$

$$=(x-\frac{3}{a})\sqrt{ax^2+2x+1}+\frac{3}{a\sqrt{a}}\int\frac{dx}{\sqrt{(x+\frac{1}{a})^2+\frac{1}{a}-\frac{1}{a^2}}}=$$

$$=(x-\frac{3}{a})\sqrt{ax^2+2x+1}+\frac{3}{a\sqrt{a}}\ln\left|x+\frac{1}{a}+\sqrt{x^2+\frac{2}{a}x+\frac{1}{a}}\right|+C$$

17.86

$$\int\frac{2x^2+3x+1}{\sqrt{x^2+1}}dx=\int\frac{2(x^2+1)}{\sqrt{x^2+1}}dx+\int\frac{3x-1}{\sqrt{x^2+1}}dx=$$
$$=2\int\sqrt{x^2+1}dx+\int\frac{\frac{3}{2}(x^2+1)'}{\sqrt{x^2+1}}dx-\int\frac{dx}{\sqrt{x^2+1}}=$$
$$=x\sqrt{x^2+1}+\ln|x+\sqrt{x^2+1}|+3\sqrt{x^2+1}-\ln|x+\sqrt{x^2+1}|+C=$$
$$=(x+3)\sqrt{x^2+1}+C$$

17.87

$$\int\frac{2x^2-ax+a^2}{\sqrt{x^2+a^2}}dx, a\neq 0$$
$$=\int\frac{2(x^2+a^2)}{\sqrt{x^2+a^2}}-\int\frac{ax+a^2}{\sqrt{x^2+a^2}}dx=2\int\sqrt{x^2+a^2}dx-a\int\frac{\frac{1}{2}(x^2+a^2)'+a}{\sqrt{x^2+a^2}}dx=$$
$$=x\sqrt{x^2+a^2}+a^2\ln|x+\sqrt{x^2+a^2}|-a\sqrt{x^2+a^2}-a^2\int\frac{dx}{\sqrt{x^2+a^2}}=$$
$$=x\sqrt{x^2+a^2}+a^2\ln|x+\sqrt{x^2+a^2}|-a\sqrt{x^2+a^2}-a^2\ln|x+\sqrt{x^2+a^2}|+C=$$
$$=(x-a)\sqrt{x^2+a^2}+C$$

17.88

$$\int\frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx \equiv (ax^2+bx+c)\sqrt{x^2+2x+2}+K\int \frac{dx}{\sqrt{x^2+2x+2}}$$

$$\frac{x^3-x+1}{\sqrt{x^2+2x+2}} \equiv (2ax+bx)\sqrt{x^2+2x+2}+\frac{(ax^2+bx+c)(x+1)}{\sqrt{x^2+2x+2}}+\frac{K}{\sqrt{x^2+2x+2}}$$

$$x^3-x+1 \equiv (2ax+bx)(x^2+2x+2)+(ax^2+bx+c)(x+1)+K$$

$$x^3-x+1 \equiv 3ax^3+(5a+2b)x^2+(4a+3b+c)x+(2b+c+K)$$

$$\begin{cases} 3a=1 \\ 5a+2b=0 \\ 4a+3b+c=-1 \\ 2b+c+K=1 \end{cases}$$

$$\begin{cases} a=\frac{1}{3} \\ b=-\frac{5}{6} \\ c=\frac{1}{6} \\ K=\frac{5}{2} \end{cases}$$

$$\int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx = (\frac{1}{3}x^2-\frac{5}{6}x+\frac{1}{6})\sqrt{x^2+2x+2}+\frac{5}{2}\int \frac{dx}{\sqrt{x^2+2x+2}} =$$

$$= (\frac{1}{3}x^2-\frac{5}{6}x+\frac{1}{6})\sqrt{x^2+2x+2}+\frac{5}{2}\ln|x+1+\sqrt{x^2+2x+2}|+C$$

17.89

$$\int \frac{x^3+2x^2+x-1}{\sqrt{x^2+2x-1}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{x^3+2x^2+x-1}{\sqrt{x^2+2x-1}}dx \equiv (ax^2+bx+c)\sqrt{x^2+2x-1}+K\int \frac{dx}{\sqrt{x^2+2x-1}}$$

$$\frac{x^3+2x^2+x-1}{\sqrt{x^2+2x-1}} \equiv (2ax+bx)\sqrt{x^2+2x-1}+\frac{(ax^2+bx+c)(x+1)}{\sqrt{x^2+2x-1}}+\frac{K}{\sqrt{x^2+2x-1}}$$

$$x^3-x+1 \equiv (2ax+bx)(x^2+2x-1)+(ax^2+bx+c)(x+1)+K$$

$$x^3-x+1 \equiv 3ax^3+(5a+2b)x^2+(-2a+3b+c)x+(-b+c+K)$$

$$\begin{cases} 3a=1 \\ 5a+2b=2 \\ -2a+3b+c=1 \\ -b+c+K=-1 \end{cases}$$

$$\begin{cases} a=\frac{1}{3} \\ b=\frac{1}{6} \\ c=\frac{7}{6} \\ K=-2 \end{cases}$$

$$\int \frac{x^3+2x^2+x-1}{\sqrt{x^2+2x-1}}dx = (\frac{1}{3}x^2+\frac{1}{6}x+\frac{7}{6})\sqrt{x^2+2x-1}-2\int \frac{dx}{\sqrt{x^2+2x-1}} =$$

$$= (\frac{1}{3}x^2 + \frac{1}{6}x + \frac{7}{6})\sqrt{x^2+2x-1} - 2\ln|x+1+\sqrt{x^2+2x-1}| + C$$

17.90
$$\int \frac{x^3dx}{\sqrt{x^2-4x+3}}$$
metoda współczynników nieoznaczonych
$$\int \frac{x^3dx}{\sqrt{x^2-4x+3}} \equiv (ax^2+bx+c)\sqrt{x^2-4x+3} + K\int \frac{dx}{\sqrt{x^2-4x+3}}$$
$$\frac{x^3}{\sqrt{x^2-4x+3}} \equiv (2ax+b)\sqrt{x^2-4x+3} + \frac{(ax^2+bx+c)(x-2)}{\sqrt{x^2-4x+3}} + \frac{K}{\sqrt{x^2-4x+3}}$$
$$x^3 \equiv (2ax+b)(x^2-4x+3) + (ax^2+bx+c)(x-2) + K$$
$$x^3 \equiv 3ax^3 + (-10a+2b)x^2 + (6a-6b+c)x + (3b-2c+K)$$
$$\begin{cases} 3a = 1 \\ -10a+2b = 0 \\ 6a-6b+c = 0 \\ 3b-2c+K = 0 \end{cases}$$
$$\begin{cases} a = \frac{1}{3} \\ b = \frac{5}{3} \\ c = 8 \\ K = 11 \end{cases}$$
$$\int \frac{x^3dx}{\sqrt{x^2-4x+3}} = (\frac{1}{3}x^2 + \frac{5}{3}x + 8)\sqrt{x^2-4x+3} + 11\int \frac{dx}{\sqrt{x^2-4x+3}} =$$
$$= (\frac{1}{3}x^2 + \frac{5}{3}x + 8)\sqrt{x^2-4x+3} + 11\ln|x-2+\sqrt{x^2-4x+3}| + C$$

17.91
$$\int \frac{3x^3+2}{\sqrt{x^2+x+1}}dx$$
metoda współczynników nieoznaczonych
$$\int \frac{3x^3+2}{\sqrt{x^2+x+1}}dx \equiv (ax^2+bx+c)\sqrt{x^2+x+1} + K\int \frac{dx}{\sqrt{x^2+x+1}}$$
$$\frac{3x^3+2}{\sqrt{x^2+x+1}} \equiv (2ax+b)\sqrt{x^2+x+1} + \frac{(ax^2+bx+c)(x+\frac{1}{2})}{\sqrt{x^2+x+1}} + \frac{K}{\sqrt{x^2+x+1}}$$
$$3x^3+2 \equiv (2ax+b)(x^2+x+1) + (ax^2+bx+c)(x+\frac{1}{2}) + K$$
$$\begin{cases} 3a = 3 \\ \frac{5}{2}a + 2b = 0 \\ 2a + \frac{3}{2}b + \frac{1}{2}c = 0 \\ b + \frac{1}{2}c + K = 2 \end{cases}$$
$$\begin{cases} a = 1 \\ b = -\frac{5}{4} \\ c = -\frac{1}{8} \\ K = \frac{53}{16} \end{cases}$$

$$\int \frac{3x^3+2}{\sqrt{x^2+x+1}}dx = (x^2-\frac{5}{4}x-\frac{1}{8})\sqrt{x^2+x+1}+\frac{53}{16}\int \frac{dx}{\sqrt{x^2+x+1}} =$$
$$= \int \frac{3x^3+2}{\sqrt{x^2+x+1}}dx = (x^2-\frac{5}{4}x-\frac{1}{8})\sqrt{x^2+x+1}+\frac{53}{16}\int \ln|x+\frac{1}{2}+\sqrt{x^2+x+1}|+C$$

17.92

$$\int x^2\sqrt{4x-x^2}dx = \int \frac{x^2(4x-x^2)}{\sqrt{4x-x^2}}dx = \int \frac{-x^4+4x^3}{\sqrt{4x-x^2}}$$

metoda współczynników nieoznaczonych

$$\int \frac{-x^4+4x^3}{\sqrt{4x-x^2}} \equiv (ax^3+bx^2+cx+d)\sqrt{4x-x^2}+K\int \frac{dx}{\sqrt{4x-x^2}}$$
$$\frac{-x^4+4x^3}{\sqrt{4x-x^2}} \equiv (3ax^2+2bx+c)\sqrt{4x-x^2}+\frac{(ax^3+bx^2+cx+d)(2-x)}{\sqrt{4x-x^2}}+\frac{K}{\sqrt{4x-x^2}}$$
$$-x^4+4x^3 \equiv (3ax^2+2bx+c)(2-x)+(ax^3+bx^2+cx+d)(2-x)+K$$

$$\begin{cases} -4a=-1 \\ 14a-3b=4 \\ 10b-2c=0 \\ 6c-d=0 \\ 2d+K=0 \end{cases}$$

$$\begin{cases} a=\frac{1}{4} \\ b=-\frac{1}{6} \\ c=-\frac{5}{6} \\ d=-5 \\ K=10 \end{cases}$$

$$\int \frac{-x^4+4x^3}{\sqrt{4x-x^2}} = (\frac{1}{4}x^3-\frac{1}{6}x^2-\frac{5}{6}x+d)\sqrt{4x-x^2}+10\int \frac{dx}{\sqrt{4x-x^2}} =$$

$$= (\frac{1}{4}x^3-\frac{1}{6}x^2-\frac{5}{6}x-5)\sqrt{4x-x^2}+10\int \frac{dx}{\sqrt{2^2+(x-2)^2}} =$$

$$= (\frac{1}{4}x^3-\frac{1}{6}x^2-\frac{5}{6}x-5)\sqrt{4x-x^2}+10\arcsin\left(\frac{x-2}{2}\right)+C$$

17.93

$$\int x\sqrt{6+x-x^2}dx = \int \frac{x(6+x-x^2)}{\sqrt{6+x-x^2}}dx = \int \frac{-x^3+x^2+6x}{\sqrt{6+x-x^2}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{-x^3+x^2+6x}{\sqrt{6+x-x^2}}dx \equiv (ax^2+bx+c)\sqrt{6+x-x^2}+K\int \frac{dx}{\sqrt{6+x-x^2}}$$
$$\frac{-x^3+x^2+6x}{\sqrt{6+x-x^2}} \equiv (2ax+b)\sqrt{6+x-x^2}+\frac{ax^2+bx+c)(\frac{1}{2}-x)}{\sqrt{6+x-x^2}}+\frac{K}{\sqrt{6+x-x^2}}$$

$$-x^3+x^2+6x \equiv (2ax+b)(6+x-x^2)+(ax^2+bx+c)(\frac{1}{2}-x)+K$$

$$\begin{cases} -3a=-1 \\ \frac{5}{2}a-2b=1 \\ 12a+\frac{3}{2}b-c=6 \\ 6b+\frac{1}{2}c+K=0 \end{cases}$$

$$\begin{cases} a=\frac{1}{3} \\ b=-\frac{1}{12} \\ c=-\frac{17}{8} \\ K=\frac{25}{16} \end{cases}$$

$$\int \frac{-x^3+x^2+6x}{\sqrt{6+x-x^2}}dx \equiv (\frac{1}{3}x^2-\frac{1}{12}x-\frac{17}{8})\sqrt{6+x-x^2}+\frac{25}{16}\int \frac{dx}{\sqrt{6+x-x^2}} =$$

$$= (\frac{1}{3}x^2-\frac{1}{12}x-\frac{17}{8})\sqrt{6+x-x^2}+\frac{25}{16}\int \frac{dx}{\sqrt{(\frac{5}{2})^2-(x-\frac{1}{2})^2}} =$$

$$= (\frac{1}{3}x^2-\frac{1}{12}x-\frac{17}{8})\sqrt{6+x-x^2}+\frac{25}{16}\arcsin\left(\frac{2x-1}{5}\right)+C$$

17.94

$$\int \frac{x^4dx}{\sqrt{5x^2+4}}$$

metoda współczynników nieoznaczonych

$$\int \frac{x^4dx}{\sqrt{5x^2+4}} \equiv (ax^3+bx^2+cx+d)\sqrt{5x^2+4}+K\int \frac{dx}{\sqrt{5x^2+4}}$$

$$\frac{x^4}{\sqrt{5x^2+4}} \equiv (3ax^2+2bx+c)\sqrt{5x^2+4}+\frac{(ax^3+bx^2+cx+d)\cdot 5x}{\sqrt{5x^2+4}}+\frac{K}{\sqrt{5x^2+4}}$$

$$x^4 \equiv (3ax^2+2bx+c)(5x^2+4)+5x(ax^3+bx^2+cx+d)+K$$

$$\begin{cases} 20a=1 \\ 15b=0 \\ 12a+10c=0 \\ 8b+5d=0 \\ 4c+K=0 \end{cases}$$

$$\begin{cases} a=\frac{1}{20} \\ b=0 \\ c=-\frac{3}{50} \\ d=0 \\ K=\frac{6}{25} \end{cases}$$

$$\int \frac{x^4dx}{\sqrt{5x^2+4}} = (\frac{1}{20}x^3-\frac{3}{50}x)\sqrt{5x^2+4}+\frac{6}{25}\int \frac{dx}{\sqrt{5x^2+4}} =$$

$$= (\frac{1}{20}x^3-\frac{3}{50}x)\sqrt{5x^2+4}+\frac{6}{25\sqrt{5}}\int \frac{dx}{\sqrt{x^2+\frac{4}{5}}} =$$

$$= (\frac{1}{20}x^3 - \frac{3}{50}x)\sqrt{5x^2+4} + \frac{6\sqrt{5}}{125}\ln|x + \sqrt{x^2+\frac{4}{5}}| + C$$

17.95

$$\int \frac{x^3+5x^2-3x+4}{\sqrt{x^2+x+1}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{x^3+5x^2-3x+4}{\sqrt{x^2+x+1}}dx \equiv (ax^2+bx+c)\sqrt{x^2+x+1} + K\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$\frac{x^3+5x^2-3x+4}{\sqrt{x^2+x+1}} \equiv (2ax+b)\sqrt{x^2+x+1} + \frac{(ax^2+bx+c)(x+\frac{1}{2})}{\sqrt{x^2+x+1}} + \frac{K}{\sqrt{x^2+x+1}}$$

$$x^3+5x^2-3x+4 \equiv (2ax+b)(x^2+x+1) + (ax^2+bx+c)(x+\frac{1}{2}) + K$$

$$\begin{cases} 3a = 1 \\ \frac{5}{2}a + 2b = 5 \\ 2a + \frac{3}{2}b + c = -3 \\ b + \frac{1}{2}c + K4 \end{cases}$$

$$\begin{cases} a = \frac{1}{3} \\ b = \frac{25}{12} \\ c = -\frac{163}{24} \\ K = \frac{85}{16} \end{cases}$$

$$\int \frac{x^3+5x^2-3x+4}{\sqrt{x^2+x+1}}dx = (\frac{1}{3}x^2 + \frac{25}{12}x - \frac{163}{24})\sqrt{x^2+x+1} + \frac{85}{16}\int \frac{dx}{\sqrt{x^2+x+1}} =$$

$$= (\frac{1}{3}x^2 + \frac{25}{12}x - \frac{163}{24})\sqrt{x^2+x+1} + \frac{85}{16}\ln|x + \frac{1}{2} + \sqrt{x^2+x+1}| + C$$

17.96

$$\int \frac{5x^2-2x+10}{\sqrt{3x^2-5x+8}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{5x^2-2x+10}{\sqrt{3x^2-5x+8}}dx \equiv (ax+b)\sqrt{3x^2-5x+8} + K\int \frac{dx}{\sqrt{3x^2-5x+8}}$$

$$\frac{5x^2-2x+10}{\sqrt{3x^2-5x+8}} \equiv a\sqrt{3x^2-5x+8} + \frac{(ax+b)(3x-\frac{5}{2})}{\sqrt{3x^2-5x+8}} + \frac{K}{\sqrt{3x^2-5x+8}}$$

$$5x^2-2x+10 \equiv a(3x^2-5x+8) + (ax+b)(3x-\frac{5}{2}) + K$$

$$\begin{cases} 6a = 5 \\ -\frac{15}{2} + 3b = -2 \\ 8a - \frac{5}{2}b + K = 10 \end{cases}$$

$$\begin{cases} a = \frac{5}{6} \\ b = \frac{17}{12} \\ K = \frac{55}{8} \end{cases}$$

$$\int \frac{5x^2-2x+10}{\sqrt{3x^2-5x+8}}dx = (\frac{5}{6}x+\frac{17}{12})\sqrt{3x^2-5x+8}+\frac{55}{8}\int \frac{dx}{\sqrt{3x^2-5x+8}} =$$

$$= (\frac{5}{6}x+\frac{17}{12})\sqrt{3x^2-5x+8}+\frac{55}{8\sqrt{3}}\int \frac{dx}{\sqrt{x^2-\frac{5}{3}x+\frac{8}{3}}} =$$

$$= (\frac{5}{6}x+\frac{17}{12})\sqrt{3x^2-5x+8}+\frac{55\sqrt{3}}{24}\ln|x-\frac{5}{6}+\sqrt{x^2-\frac{5}{3}x+\frac{8}{3}}|+C$$

17.97

$$\int \frac{x^3+4x^2-6x+3}{\sqrt{5+6x-x^2}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{x^3+4x^2-6x+3}{\sqrt{5+6x-x^2}}dx \equiv (ax^2+bx+c)\sqrt{5+6x-x^2}+K\int \frac{dx}{\sqrt{5+6x-x^2}}$$

$$\frac{x^3+4x^2-6x+3}{\sqrt{5+6x-x^2}} \equiv (2ax+b)\sqrt{5+6x-x^2}+\frac{(ax^2+bx+c)(3-x)}{\sqrt{5+6x-x^2}}+\frac{K}{\sqrt{5+6x-x^2}}$$

$$x^3+4x^2-6x+3 \equiv (2ax+b)(5+6x-x^2)+(ax^2+bx+c)(3-x)+K$$

$$\begin{cases} -3a=1 \\ 15a-2b=4 \\ 10a+9b-c=-6 \\ 5b+3c+K=3 \end{cases}$$

$$\begin{cases} a=-\frac{1}{3} \\ b=-\frac{9}{2} \\ c=-\frac{227}{6} \\ K=139 \end{cases}$$

$$\int \frac{x^3+4x^2-6x+3}{\sqrt{5+6x-x^2}}dx = (-\frac{1}{3}x^2-\frac{9}{2}x-\frac{227}{6})\sqrt{5+6x-x^2}+139\int \frac{dx}{\sqrt{5+6x-x^2}} =$$

$$= (-\frac{1}{3}x^2-\frac{9}{2}x-\frac{227}{6})\sqrt{5+6x-x^2}+139\int \frac{dx}{\sqrt{(\sqrt{14})^2+(x-3)^2}} =$$

$$= (-\frac{1}{3}x^2-\frac{9}{2}x-\frac{227}{6})\sqrt{5+6x-x^2}+139\arcsin\left(\frac{x-3}{\sqrt{14}}\right)+C$$

17.98

$$\int x\sqrt{8+x-x^2}dx = \int \frac{x(8+x-x^2)}{\sqrt{8+x-x^2}}dx = \int \frac{-x^3+x^2+8x}{\sqrt{8+x-x^2}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{-x^3+x^2+8x}{\sqrt{8+x-x^2}}dx \equiv (ax^2+bx+c)\sqrt{8+x-x^2}+K\int \frac{dx}{\sqrt{8+x-x^2}}$$

$$\frac{-x^3+x^2+8x}{\sqrt{8+x-x^2}} \equiv (2ax+b)\sqrt{8+x-x^2} + \frac{(ax^2+bx+c)(\frac{1}{2}-x)}{\sqrt{8+x-x^2}} + \frac{K}{\sqrt{8+x-x^2}}$$

$$-x^3+x^2+8x \equiv (2ax+b)(8+x-x^2) + (ax^2+bx+c)(\frac{1}{2}-x) + K$$

$$\begin{cases} -3a=-1 \\ \frac{5}{2}a-2b=1 \\ 16a+\frac{3}{2}b-c=8 \\ 8b+\frac{1}{2}c+K=0 \end{cases}$$

$$\begin{cases} a=\frac{1}{3} \\ b=-\frac{1}{12} \\ c=-\frac{67}{24} \\ K=\frac{33}{16} \end{cases}$$

$$\int \frac{-x^3+x^2+8x}{\sqrt{8+x-x^2}}dx = (\frac{1}{3}x^2-\frac{1}{12}x-\frac{67}{24})\sqrt{8+x-x^2} + \frac{33}{16}\int \frac{dx}{\sqrt{(\frac{\sqrt{33}}{2})^2-(x-\frac{1}{2})^2}} =$$

$$= (\frac{1}{3}x^2-\frac{1}{12}x-\frac{67}{24})\sqrt{8+x-x^2} + \frac{33}{16}\arcsin\left(\frac{2x-1}{\sqrt{33}}\right) + C$$

17.99

$$\int (2x-5)\sqrt{2+3x-x^2}dx = \int \frac{(2x-5)(2+3x-x^2)}{\sqrt{2+3x-x^2}}dx = \int \frac{-2x^3+11x^2-11x-10}{\sqrt{2+3x-x^2}}dx$$

metoda współczynników nieoznaczonych

$$\int \frac{-2x^3+11x^2-11x-10}{\sqrt{2+3x-x^2}}dx \equiv (ax^2+bx+c)\sqrt{2+3x-x^2} + K\int \frac{dx}{\sqrt{2+3x-x^2}}$$

$$\frac{-2x^3+11x^2-11x-10}{\sqrt{2+3x-x^2}} \equiv (2ax+b)\sqrt{2+3x-x^2} + \frac{(ax^2+bx+c)(\frac{3}{2}-x)}{\sqrt{2+3x-x^2}} + \frac{K}{\sqrt{2+3x-x^2}}$$

$$-2x^3+11x^2-11x-10 \equiv (2ax+b)(2+3x-x^2) + (ax^2+bx+c)(\frac{3}{2}-x) + K$$

$$\begin{cases} -3a=-2 \\ \frac{15}{2}a-2b=11 \\ 4a+\frac{9}{2}b-c=-11 \\ 2b+\frac{3}{2}c+K=-10 \end{cases}$$

$$\begin{cases} a=\frac{2}{3} \\ b=-3 \\ c=\frac{1}{6} \\ K=-\frac{17}{4} \end{cases}$$

$$\int \frac{-2x^3+11x^2-11x-10}{\sqrt{2+3x-x^2}}dx = (\frac{2}{3}x^2-3x+\frac{1}{6})\sqrt{2+3x-x^2} - \frac{17}{4}\int \frac{dx}{\sqrt{2+3x-x^2}} =$$

$$= (\frac{2}{3}x^2-3x+\frac{1}{6})\sqrt{2+3x-x^2} - \frac{17}{4}\int \frac{dx}{\sqrt{(\frac{\sqrt{17}}{2})^2-(x-\frac{3}{2})^2}} =$$

$$= (\frac{2}{3}x^2 - 3x + \frac{1}{6})\sqrt{2+3x-x^2} - \frac{17}{4}\arcsin\left(\frac{2x-3}{\sqrt{17}}\right) + C$$

17.100

$$\int \frac{x^3dx}{\sqrt{2x^2+3}}$$

metoda współczynników nieoznaczonych

$$\int \frac{x^3dx}{\sqrt{2x^2+3}} \equiv (ax^2+bx+c)\sqrt{2x^2+3} + K\int \frac{dx}{\sqrt{2x^2+3}}$$

$$\frac{x^3}{\sqrt{2x^2+3}} \equiv (2ax+b)\sqrt{2x^2+3} + \frac{(ax^2+bx+c)\cdot 2x}{\sqrt{2x^2+3}} + \frac{K}{\sqrt{2x^2+3}}$$

$$x^3 \equiv (2ax+b)(2x^2+3) + 2x(ax^2+bx+c) + K$$

$$\begin{cases} 6a = 1 \\ 4b = 0 \\ 6a + 2c = 0 \\ 3b + K = 0 \end{cases}$$

$$\begin{cases} a = \frac{1}{6} \\ b = 0 \\ c = -\frac{1}{2} \\ K = 0 \end{cases}$$

$$\int \frac{x^3dx}{\sqrt{2x^2+3}} = (\frac{1}{6}x^2 - \frac{1}{2})\sqrt{2x^2+3} + C$$

17.101

$$\int \frac{x^5dx}{\sqrt{2x^2+3}}$$

metoda współczynników nieoznaczonych

$$\int \frac{x^5dx}{\sqrt{2x^2+3}} \equiv (ax^4+bx^3+cx^2+dx+e)\sqrt{2x^2+3} + K\int \frac{dx}{\sqrt{2x^2+3}}$$

$$\frac{x^5}{\sqrt{2x^2+3}} \equiv (4ax^3+3bx^2+2cx+d)\sqrt{2x^2+3} + \frac{(ax^4+bx^3+cx^2+dx+e)\cdot 2x}{\sqrt{2x^2+3}} + \frac{K}{\sqrt{2x^2+3}}$$

$$x^5 \equiv (4ax^3+3bx^2+2cx+d)(2x^2+3) + 2x(ax^4+bx^3+cx^2+dx+e) + K$$

$$\begin{cases} 10a = 1 \\ 8b = 0 \\ 12a + 6c = 0 \\ 9b + 4d = 0 \\ 3d + K = 0 \end{cases}$$

$$\begin{cases} a = \frac{1}{10} \\ b = 0 \\ c = -\frac{1}{5} \\ d = 0 \\ e = \frac{3}{5} \\ K = 0 \end{cases}$$

$$\int \frac{x^5 dx}{\sqrt{2x^2+3}} = (\frac{1}{10}x^4 - \frac{1}{5}x^2 + \frac{3}{5})\sqrt{2x^2+3} + C$$

17.102

$$\int \frac{x^4 dx}{\sqrt{3+2x+x^2}}$$

metoda współczynników nieoznaczonych

$$\int \frac{x^4 dx}{\sqrt{3+2x+x^2}} \equiv (ax^3+bx^2+cx+d)\sqrt{3+2x+x^2} + K\int \frac{dx}{\sqrt{3+2x+x^2}}$$

$$\frac{x^4}{\sqrt{3+2x+x^2}} \equiv (3ax^2+2bx+c)\sqrt{3+2x+x^2} + \frac{(ax^3+bx^2+cx+d)(x+1)}{\sqrt{3+2x+x^2}} + \frac{K}{\sqrt{3+2x+x^2}}$$

$$x^4 \equiv (3ax^2+2bx+c)(x^2+2x+3) + (ax^3+bx^2+cx+d)(x+1) + K$$

$$\begin{cases} 4a = 1 \\ 7a + 3b = 0 \\ 9a + 5b + 2c = 0 \\ 6b + 3c + d = 0 \\ 3c + d + K = 0 \end{cases}$$

$$\begin{cases} a = \frac{1}{4} \\ b = -\frac{7}{12} \\ c = \frac{1}{3} \\ d = \frac{5}{2} \\ K = -\frac{7}{2} \end{cases}$$

$$\int \frac{x^4 dx}{\sqrt{3+2x+x^2}} \equiv (\frac{1}{4}x^3 - \frac{7}{12}x^2 + \frac{1}{3}x + \frac{5}{2})\sqrt{3+2x+x^2} - \frac{7}{2}\int \frac{dx}{\sqrt{3+2x+x^2}} =$$

$$= (\frac{1}{4}x^3 - \frac{7}{12}x^2 + \frac{1}{3}x + \frac{5}{2})\sqrt{3+2x+x^2} - \frac{7}{2}\ln|x+1+\sqrt{3+2x+x^2}| + C$$

17.103

$$\int \frac{dx}{x\sqrt{10x-x^2}} = \left| \begin{array}{c} t = \frac{1}{x} \\ \frac{1}{t} = x \\ -\frac{dt}{t^2} = dx \end{array} \right| = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{10}{t} - \frac{1}{t^2}}} = \int \frac{-dt}{t\sqrt{\frac{10}{t} - \frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{10t-1}} =$$

$$= -\int \frac{\frac{1}{10}(10t-1)'}{\sqrt{10t-1}}dt = -\frac{1}{5}\sqrt{10t-1} + C = -\frac{1}{5}\sqrt{\frac{10}{x} - 1} + C$$

$$\int \frac{\mathrm{d}x}{x\sqrt{10x-x^2}} = \left| \begin{array}{c} \sqrt{10x-x^2} = xt \\ 10x - x^2 = x^2t^2 \\ 10 - x = xt^2 \\ x + xt^2 = 10 \\ x(1+t^2) = 10 \\ x = \dfrac{10}{1+t^2} \\ \mathrm{d}x = \dfrac{-20t}{(1+t^2)^2}\mathrm{d}t \\ \sqrt{10x-x^2} = \dfrac{10t}{1+t^2} \end{array} \right| = \int \frac{1+t^2}{10}\cdot\frac{1+t^2}{10t}\cdot\frac{-20t}{(1+t^2)^2}\mathrm{d}t =$$

$$= -\frac{1}{5}\int \mathrm{d}t = -\frac{1}{5}t + C = -\frac{1}{5}\frac{\sqrt{10x-x^2}}{x} + C$$

17.104

$$\int \frac{\mathrm{d}x}{(x+1)\sqrt{x^2-1}} = \left| \begin{array}{c} \sqrt{x^2-1} = (x+1)t \\ x^2 - 1 = (x+1)^2t^2 \\ x - 1 = (x+1)t^2 \\ x - 1 = xt^2 + t^2 \\ x - xt^2 = 1 + t^2 \\ x(1-t^2) = 1+t^2 \\ x = \dfrac{1+t^2}{1-t^2} \\ x = -1 + \dfrac{2}{1-t^2} \\ \mathrm{d}x = \dfrac{4t}{(1-t^2)^2}\mathrm{d}t \\ x + 1 = \dfrac{2}{1-t^2} \\ \sqrt{x^2-1} = \dfrac{2t}{1-t^2} \end{array} \right| = \int \frac{1-t^2}{2}\cdot\frac{1-t^2}{2t}\cdot\frac{4t}{(1-t^2)^2}\mathrm{d}t =$$

$$= \int \mathrm{d}t = t + C = \frac{\sqrt{x^2-1}}{x+1} + C$$

17.105

$$\int \frac{\mathrm{d}x}{(x+2)\sqrt{4-x^2}} = \left| \begin{array}{c} \sqrt{4-x^2} = (x+2)t \\ 4 - x^2 = (x+2)^2t^2 \\ 2 - x = (x+2)t^2 \\ 2 - x = xt^2 + 2t^2 \\ x + xt^2 = 2 - 2t^2 \\ x(1+t^2) = 2 - 2t^2 \\ x = \dfrac{2-2t^2}{t^2+1} \\ x = -2 + \dfrac{4}{1+t^2} \\ \mathrm{d}x = -\dfrac{8t}{(1+t^2)^2} \\ x + 2 = \dfrac{4}{1+t^2} \\ \sqrt{4-x^2} = \dfrac{4t}{1+t^2} \end{array} \right| = \int \frac{1+t^2}{4}\cdot\frac{1+t^2}{4t}\cdot\frac{-8t}{(1+t^2)^2}\mathrm{d}t =$$

$$= -\frac{1}{2}\int \mathrm{d}t = -\frac{1}{2}t + C = -\frac{1}{2}\cdot\frac{\sqrt{4-x^2}}{2+x} + C$$

17.106

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2+x-1}} = \left|\begin{array}{c} \sqrt{x^2+x-1} = t-x \\ x^2+x-1 = t^2-2tx+x^2 \\ x-1 = t^2-2tx \\ 2tx+x = t^2+1 \\ x(2t+1) = t^2+1 \\ x = \dfrac{t^2+1}{2t+1} \\ \mathrm{d}x = \dfrac{2t\cdot(2t+1)-2(t^2+1)}{(2t+1)^2}\mathrm{d}t \\ \mathrm{d}x = \dfrac{2t^2+2t-2}{(2t+1)^2}\mathrm{d}t \\ \sqrt{x^2+x-1} = t - \dfrac{t^2+1}{2t+1} \\ \sqrt{x^2+x-1} = \dfrac{t^2+t-1}{2t+1} \end{array}\right| = \int \frac{2t+1}{t^2+1}\cdot\frac{2t+1}{t^2+t-1}\cdot\frac{2(t^2+t-1)}{(2t+1)^2}\mathrm{d}t =$$

$$= 2\int \frac{\mathrm{d}t}{t^2+1} = 2\arctan t + C =$$

$$= 2\arctan\left(x+\sqrt{x^2+x-1}\right) + C$$

17.107

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-2x-1}} = \left|\begin{array}{c} \sqrt{x^2-2x-1} = t-x \\ x^2-2x-1 = t^2-2tx+x^2 \\ -2x-1 = t^2-2tx \\ t^2+1 = 2tx-2x \\ x(2t-2) = t^2+1 \\ x = \dfrac{t^2+1}{2t-2} \\ \mathrm{d}x = \dfrac{2t(2t-2)-2(t^2+1)}{(2t-2)^2}\mathrm{d}t \\ \mathrm{d}x = \dfrac{2t^2-4t-2}{(2t-2)^2}\mathrm{d}t \\ \sqrt{x^2-2x-1} = t - \dfrac{t^2+1}{2t-2} \\ \sqrt{x^2-2x-1} = \dfrac{t^2-2t-1}{2t-2} \end{array}\right| = \int \frac{2t-2}{t^2+1}\cdot\frac{2t-2}{t^2-2t-1}\cdot\frac{2(t^2-2t-1)}{(2t-2)^2}\mathrm{d}t =$$

$$= 2\int \frac{\mathrm{d}t}{t^2+1} = 2\arctan t + C =$$

$$= 2\arctan\left(x+\sqrt{x^2-2x-1}\right) + C$$

17.108

$$\int \frac{\mathrm{d}x}{(2x-1)\sqrt{x^2-1}} = \left| \begin{array}{c} \sqrt{x^2-1} = t - x \\ x^2 - 1 = t^2 - 2tx + x^2 \\ -1 = t^2 - 2tx \\ 2tx = t^2 + 1 \\ x = \frac{t^2+1}{2t} \\ \mathrm{d}x = \frac{t^2-1}{2t^2}\mathrm{d}t \\ \sqrt{x^2-1} = t - \frac{t^2+1}{2t} \\ \sqrt{x^2-1} = \frac{t^2-1}{2t} \\ 2x - 1 = \frac{t^2-t+1}{t} \end{array} \right| = \int \frac{t}{t^2-t+1} \cdot \frac{2t}{t^2-1} \cdot \frac{t^2-1}{2t^2}\mathrm{d}t =$$

$$= \int \frac{\mathrm{d}t}{t^2-t+1} = \int \frac{\mathrm{d}t}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}} =$$

$$= \frac{4}{3}\int \frac{\mathrm{d}t}{\left(\frac{2t-1}{\sqrt{3}}\right)^2 + 1} = \frac{2}{\sqrt{3}} \arctan\left(\frac{2t-1}{\sqrt{3}}\right) + C =$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2x + 2\sqrt{x^2-1} - 1}{\sqrt{3}}\right) + C$$

17.109

$$\int \frac{\mathrm{d}x}{(x+1)\sqrt{1+2x-3x^2}} = \left| \begin{array}{c} \sqrt{1+2x-3x^2} = xt + 1 \\ 1 + 2x - 3x^2 = x^2t^2 + 2xt + 1 \\ 2x - 3x^2 = x^2t^2 + 2xt \\ 2 - 3x = xt^2 + 2t \\ 2 - 2t = xt^2 + 3x \\ x(t^2+3) = 2 - 2t \\ x = \frac{2-2t}{t^2+3} \\ \mathrm{d}x = \frac{2(t^2-2t-3)}{(t^2+3)^2}\mathrm{d}t \\ x + 1 = \frac{t^2-2t+5}{t^2+3} \\ \sqrt{1+2x-3x^2} = \frac{2-2t}{t^2+3} \cdot t + 1 \\ \sqrt{1+2x-3x^2} = \frac{-(t^2-2t-3)}{t^2+3} \end{array} \right|$$

$$= \int \frac{t^2+3}{t^2-2t+5} \cdot \frac{t^2+3}{-(t^2-2t-3)} \cdot \frac{2(t^2-2t-3)}{(t^2+3)^2}\mathrm{d}t = -2\int \frac{\mathrm{d}t}{t^2-2t+5} =$$

$$= -2\int \frac{\mathrm{d}t}{(t-1)^2+4} = -\frac{1}{2}\int \frac{\mathrm{d}t}{1 + \left(\frac{t-1}{2}\right)^2} = -\arctan\left(\frac{t-1}{2}\right) + C =$$

$$= -\arctan\left(\frac{\sqrt{1+2x-3x^2} - x - 1}{2x}\right) + C$$

17.110

$$\int \frac{\mathrm{d}x}{(3-2x)\sqrt{x^2-4x+3}} = \left| \begin{array}{c} \sqrt{x^2-4x+3} = t-x \\ x^2-4x+3 = t^2-2tx+x^2 \\ -4x+3 = t^2-2tx \\ 2tx-4x = t^2-3 \\ x(2t-4) = t^2-3 \\ x = \dfrac{t^2-3}{2t-4} \\ \mathrm{d}x = \dfrac{2(t^2-4t+3)}{(2t-4)^2}\mathrm{d}t \\ 3-2x = \dfrac{-2(t^2-3t+3)}{2t-4} \\ \sqrt{x^2-4x+3} = t - \dfrac{t^2-3}{2t-4} \\ \sqrt{x^2-4x+3} = \dfrac{t^2-4t+3}{2t-4} \end{array} \right| =$$

$$= \int \frac{2t-4}{-2(t^2-3t+3)} \cdot \frac{2t-4}{t^2-4t+3} \cdot \frac{2(t^2-4t+3)}{(2t-4)^2}\mathrm{d}t = -\int \frac{\mathrm{d}t}{t^2-3t+3} =$$

$$= -\int \frac{\mathrm{d}t}{\left(t-\frac{3}{2}\right)^2+\frac{3}{4}} = -\frac{4}{3}\int \frac{\mathrm{d}t}{1+\left(\frac{2t-3}{\sqrt{3}}\right)^2} = -\frac{2}{\sqrt{3}}\arctan\left(\frac{2t-3}{\sqrt{3}}\right) + C =$$

$$= -\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+2\sqrt{x^2-4x+3}-3}{\sqrt{3}}\right) + C$$

17.111

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2+x+1}} = \left| \begin{array}{c} \sqrt{x^2+x+1} = xt+1 \\ x^2+x+1 = x^2t^2+2xt+1 \\ x^2+x = x^2t^2+2xt \\ x+1 = xt^2+2t \\ x-xt^2 = 2t-1 \\ x(1-t^2) = 2t-1 \\ x = \dfrac{2t-1}{1-t^2} \\ \mathrm{d}x = \dfrac{2(t^2-t+1)}{(1-t^2)^2}\mathrm{d}t \\ \sqrt{x^2+x+1} = \dfrac{2t-1}{1-t^2}\cdot t+1 \\ \sqrt{x^2+x+1} = \dfrac{t^2-t+1}{1-t^2} \end{array} \right| = \int \frac{1-t^2}{2t-1} \cdot \frac{1-t^2}{t^2-t+1} \cdot \frac{2(t^2-t+1)}{(1-t^2)^2}\mathrm{d}t$$

$$= \int \frac{2}{2t-1}\mathrm{d}t = \ln|2t-1| + C = \ln\left|\frac{2\sqrt{x^2+x+1}-x-2}{x}\right| + C$$

17.112

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \left| \begin{array}{c} \sqrt{x^2-1} = t - x \\ x^2 - 1 = t^2 - 2tx + x^2 \\ -1 = t^2 - 2tx \\ 2tx = t^2 + 1 \\ x = \dfrac{t^2+1}{2t} \\ \mathrm{d}x = \dfrac{t^2-1}{2t^2}\mathrm{d}t \\ \sqrt{x^2-1} = t - \dfrac{t^2+1}{2t} \\ \sqrt{x^2-1} = \dfrac{t^2-1}{2t} \end{array} \right| = \int \frac{2t}{t^2+1} \cdot \frac{2t}{t^2-1} \cdot \frac{t^2-1}{2t^2}\mathrm{d}t = \int \frac{2\mathrm{d}t}{t^2+1} =$$

$$= 2\arctan t + C = 2\arctan\left(x + \sqrt{x^2-1}\right) + C$$

17.113

$$\int \frac{\mathrm{d}x}{(a-x)\sqrt{a^2-x^2}} = \left| \begin{array}{c} \sqrt{a^2-x^2} = (a-x)t \\ a^2 - x^2 = (a-x)^2t^2 \\ a + x = (a-x)t^2 \\ a + x = at^2 - xt^2 \\ x + xt^2 = at^2 - a \\ x(1+t^2) = at^2 - a \\ x = \dfrac{at^2-a}{1+t^2} \\ x = a - \dfrac{2a}{1+t^2} \\ \mathrm{d}x = \dfrac{4at}{(1+t^2)^2}\mathrm{d}t \end{array} \right| = \int \frac{1+t^2}{2a} \cdot \frac{1+t^2}{2at} \cdot \frac{4at}{(1+t^2)^2}\mathrm{d}t =$$

$$= \frac{1}{a}\int \mathrm{d}t = \frac{1}{a} \cdot t + C = \frac{1}{a} \cdot \frac{\sqrt{a^2-x^2}}{a-x} + C$$

17.114

$$\int \frac{\mathrm{d}x}{(x-2)\sqrt{x^2-6x+1}} = \left| \begin{array}{c} \sqrt{x^2-6x+1} = t - x \\ x^2 - 6x + 1 = t^2 - 2tx + x^2 \\ -6x + 1 = t^2 - 2tx \\ 2tx - 6x = t^2 - 1 \\ x(2t-6) = t^2 - 1 \\ x = \dfrac{t^2-1}{2t-6} \\ \mathrm{d}x = \dfrac{2(t^2-6t+1)}{(2t-6)^2}\mathrm{d}t \\ x - 2 = \dfrac{t^2-1}{2t-6} - 2 \\ x - 2 = \dfrac{t^2-4t+11}{2t-6} \\ \sqrt{x^2-6x+1} = t - \dfrac{t^2-1}{2t-6} \\ \sqrt{x^2-6x+1} = \dfrac{t^2-6t+1}{2t-6} \end{array} \right| = 2\int \frac{2t-6}{t^2-4t+11} \cdot \frac{2t-6}{t^2-6t+1} \cdot \frac{t^2-6t+1}{(2t-6)^2}\mathrm{d}t =$$

$$= 2\int \frac{\mathrm{d}t}{t^2-4t+11} = 2\int \frac{\mathrm{d}t}{(t-2)^2+7} = \frac{2}{7}\int \frac{\mathrm{d}t}{1+\left(\frac{t-2}{\sqrt{7}}\right)^2} =$$

$$= \frac{2}{\sqrt{7}}\arctan\left(\frac{t-2}{\sqrt{7}}\right) + C = \frac{2}{\sqrt{7}}\arctan\left(\frac{x-2+\sqrt{x^2-6x+1}}{\sqrt{7}}\right) + C$$

17.115

$$\int \frac{\mathrm{d}x}{x^2\sqrt{4-x^2}} = \left|\begin{array}{c} \sqrt{4-x^2} = (x+2)t \\ 4-x^2 = (x+2)^2t^2 \\ 2-x = (x+2)t^2 \\ 2-x = xt^2+2t^2 \\ 2-2t^2 = x+xt^2 \\ 2-2t^2 = x(1+t^2) \\ x = \frac{2-2t^2}{1+t^2} \\ \mathrm{d}x = \frac{-8t}{(1+t^2)^2}\mathrm{d}t \\ \sqrt{4-x^2} = \frac{4t}{1+t^2} \end{array}\right| = -\int \frac{(1+t^2)^2}{(2-2t^2)^2}\cdot\frac{1+t^2}{4t}\cdot\frac{8t}{(1+t^2)^2}\mathrm{d}t =$$

$$= -\frac{1}{2}\int \frac{1+t^2}{(1-t^2)^2}\mathrm{d}t = -\frac{1}{4}\int \frac{(1+t)^2+(1-t)^2}{(1-t)^2(1+t)^2}\mathrm{d}t =$$

$$= -\frac{1}{4}\left(\int \frac{\mathrm{d}t}{(1+t)^2} + \int \frac{\mathrm{d}t}{(1-t)^2}\right) = -\frac{1}{4}\left(\frac{1}{1-t} - \frac{1}{1+t}\right) + C =$$

$$= -\frac{1}{4}\cdot\frac{2t}{1-t^2} + C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

17.116

$$\int \frac{\mathrm{d}x}{(x-1)^2\sqrt{10x-x^2}} = \left|\begin{array}{c} \sqrt{10x-x^2} = xt \\ 10x-x^2 = x^2t^2 \\ 10-x = xt^2 \\ 10 = x+xt^2 \\ 10 = x(1+t^2) \\ x = \frac{10}{1+t^2} \\ \mathrm{d}x = \frac{-20t}{(1+t^2)^2}\mathrm{d}t \\ x-1 = \frac{9-t^2}{1+t^2} \\ \sqrt{10x-x^2} = \frac{10t}{1+t^2} \end{array}\right| = -2\int \frac{(1+t^2)^2}{(9-t^2)^2}\cdot\frac{1+t^2}{10t}\cdot\frac{10t}{(1+t^2)^2}\mathrm{d}t =$$

$$= -2\int \frac{1+t^2}{(9-t^2)^2}\mathrm{d}t$$

$$-2\int \frac{1+t^2}{(9-t^2)^2}\mathrm{d}t \equiv \int \frac{A}{3-t}\mathrm{d}t + \int \frac{B}{(3-t)^2}\mathrm{d}t + \int \frac{C}{3+t}\mathrm{d}t + \int \frac{D}{(3+t)^2}\mathrm{d}t$$

$$\begin{cases} -A+C=0 \\ -3A+B-3C+D=-2 \\ 9A+6B-9C-6D=0 \\ 27A+9B+27C+9D=-2 \end{cases}$$

$$\begin{cases} A = \frac{4}{27} \\ B = -\frac{5}{9} \\ C = \frac{4}{27} \\ D = -\frac{5}{9} \end{cases}$$

$$= \frac{4}{27}\int \frac{1}{3-t}\mathrm{d}t - \frac{5}{9}\int \frac{1}{(3-t)^2}\mathrm{d}t + \frac{4}{27}\int \frac{1}{3+t}\mathrm{d}t - \frac{5}{9}\int \frac{1}{(3+t)^2}\mathrm{d}t$$

$$= \frac{4}{27}\ln\left|\frac{3+t}{3-t}\right| - \frac{5}{9}\cdot\frac{1}{3-t} + \frac{5}{9}\cdot\frac{1}{3-t} + C$$

$$= \frac{4}{27}\ln\left|\frac{3+t}{3-t}\right| - \frac{10}{9}\cdot\frac{t}{9-t^2}$$

$$= -\frac{1}{9}\frac{\sqrt{10x-x^2}}{x-1} + \frac{4}{27}\ln\left|\frac{4x+5+3\sqrt{10x-x^2}}{x-1}\right| + C$$

17.117

$$\int \frac{\mathrm{d}x}{x^3\sqrt{x^2+1}} = \left| \begin{array}{c} \sqrt{x^2+1} = xt+1 \\ x^2+1 = x^2t^2+2xt+1 \\ x^2 = x^2t^2+2xt \\ x = xt^2+2t \\ x - xt^2 = 2t \\ x(1-t^2) = 2t \\ x = \dfrac{2t}{1-t^2} \\ \mathrm{d}x = \dfrac{2(1-t^2)+2t\cdot 2t}{(1-t^2)^2}\mathrm{d}t \\ \mathrm{d}x = \dfrac{2(1+t^2)}{(1-t^2)^2}\mathrm{d}t \\ \sqrt{x^2+1} = \dfrac{2t}{1-t^2}\cdot t+1 \\ \sqrt{x^2+1} = \dfrac{1+t^2}{1-t^2} \end{array} \right| = \int \frac{(1-t^2)^3}{8t^3}\cdot\frac{1-t^2}{1+t^2}\cdot\frac{2(1+t^2)}{(1-t^2)^2}\mathrm{d}t =$$

$$= \frac{1}{4}\int \frac{(1-t^2)^2}{t^3}\mathrm{d}t = \frac{1}{4}\left(\int t\mathrm{d}t - 2\int\frac{\mathrm{d}t}{t} + \int\frac{\mathrm{d}t}{t^3}\right) =$$

$$= \frac{1}{4}\left(\frac{t^2}{2} - 2\ln|t| - \frac{1}{2t^2}\right) + C = \frac{1}{8}\left(\frac{(t^2-1)(t^2+1)}{t^2} - 4\ln|t|\right) + C =$$

$$= -\frac{1}{2}\left(\frac{\sqrt{x^2+1}}{x^2} + \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right|\right) + C$$

17.118

$$\int \frac{\mathrm{d}x}{x^3\sqrt{2x^2+2x+1}} = \left| \begin{array}{c} \sqrt{2x^2+2x+1} = xt+1 \\ 2x^2+2x+1 = x^2t^2+2xt+1 \\ 2x^2+2x = x^2t^2+2xt \\ 2x+2 = xt^2+2t \\ 2x-xt^2 = 2t-2 \\ x(2-t^2) = 2t-2 \\ x = \dfrac{2t-2}{2-t^2} \\ \mathrm{d}x = \dfrac{2(2-t^2)+2t(2t-2)}{(2-t^2)^2}\mathrm{d}t \\ \mathrm{d}x = \dfrac{2t^2-4t+4}{(2-t^2)^2}\mathrm{d}t \\ \sqrt{2x^2+2x+1} = xt+1 \\ \sqrt{2x^2+2x+1} = \dfrac{2t^2-2t+2-t^2}{2-t^2} \\ \sqrt{2x^2+2x+1} = \dfrac{t^2-2t+2}{2-t^2} \end{array} \right| = \int \frac{(2-t^2)^3}{(2t-2)^3}\cdot\frac{2-t^2}{t^2-2t+2}\cdot\frac{2t^2-4t+4}{(2-t^2)^2}\mathrm{d}t$$

$$= 2\int \frac{(2-t^2)^2}{(2t-2)^3}\mathrm{d}t = \frac{1}{4}\int \frac{(2-t^2)^2}{(t-1)^3}\mathrm{d}t$$

$$= \frac{1}{4}\left(\int t\mathrm{d}t + 3\int \mathrm{d}t + \int \frac{2t^2-8t+7}{(t-1)^3}\mathrm{d}t\right)$$

$$= \frac{1}{4}\left(\int t\mathrm{d}t + 3\int \mathrm{d}t + \int \frac{2}{t-1}\mathrm{d}t - \int \frac{4}{(t-1)^2}\mathrm{d}t + \int \frac{1}{(t-1)^3}\mathrm{d}t\right)$$

$$= \frac{1}{4}\left(\frac{t^2}{2} + 3t + 2\ln|t-1| + \frac{4}{t-1} - \frac{1}{2}\cdot\frac{1}{(t-1)^2}\right) + C$$

$$= -\frac{1}{2}\left(\frac{\sqrt{2x^2+2x+1}}{x^2} - 3\frac{\sqrt{2x^2+2x+1}}{x} - \ln\left|\frac{\sqrt{2x^2+2x+1}-x-1}{x}\right|\right) + C$$

□

$$\int \frac{2t^2-8t+7}{(t-1)^3}\mathrm{d}t \equiv \int \frac{A}{t-1}\mathrm{d}t + \int \frac{B}{(t-1)^2}\mathrm{d}t + \int \frac{C}{(t-1)^3}\mathrm{d}t)$$

$$\begin{cases} A = 2 \\ -2A+B = -8 \\ A-B+C = 7 \end{cases}$$

$$\begin{cases} A = 2 \\ B = -4 \\ C = 1 \end{cases}$$

$$\int \frac{2t^2-8t+7}{(t-1)^3}\mathrm{d}t = \int \frac{2}{t-1}\mathrm{d}t + \int \frac{-4}{(t-1)^2}\mathrm{d}t + \int \frac{1}{(t-1)^3}\mathrm{d}t)$$

17.119

$$\int \frac{\mathrm{d}x}{(x-1)^3\sqrt{3-2x^2}} = \int \frac{\sqrt{3}\mathrm{d}x}{(x-1)^3\sqrt{9-6x^2}} =$$

$$\left|\begin{array}{c}\sqrt{9-6x^2}=xt+3\\ 9-6x^2=x^2t^2+6xt+9\\ -6x^2=x^2t^2+6xt\\ -6x=xt^2+6t\\ xt^2+6x=-6t\\ x(t^2+6)=-6t\\ x=\dfrac{-6t}{t^2+6}\\ \mathrm{d}x=\dfrac{-6(t^2+6)+12t^2}{(t^2+6)^2}\mathrm{d}t\\ \mathrm{d}x=\dfrac{6t^2-36}{(t^2+6)^2}\mathrm{d}t\\ x-1=\dfrac{-6t-t^2-6}{t^2+6}\\ \sqrt{9-6x^2}=\dfrac{-6t}{t^2+6}\cdot t+3\\ \sqrt{9-6x^2}=\dfrac{-3t^2+18}{t^2+6}\end{array}\right|$$

$$=\int\frac{(t^2+6)^3}{(t^2+6t+6)^3}\cdot\frac{t^2+6}{3t^2-18}\cdot\frac{6t^2-36}{(t^2+6)^2}\mathrm{d}t$$

$$=2\int\frac{(t^2+6)^2}{(t^2+6t+6)^3}\mathrm{d}t=2\int\frac{(t^2+6)^2}{(t+3-\sqrt{3})^3(t+3+\sqrt{3})^3}\mathrm{d}t$$

$$2\int\frac{(t^2+6)^2}{(t+3-\sqrt{3})^3(t+3+\sqrt{3})^3}\mathrm{d}t\equiv\int\frac{A}{t+3-\sqrt{3}}\mathrm{d}t+\int\frac{B}{(t+3-\sqrt{3})^2}\mathrm{d}t+\int\frac{C}{(t+3-\sqrt{3})^3}\mathrm{d}t$$

$$+\int\frac{D}{t+3+\sqrt{3}}\mathrm{d}t+\int\frac{E}{(t+3+\sqrt{3})^2}\mathrm{d}t+\int\frac{F}{(t+3+\sqrt{3})^3}\mathrm{d}t$$

$$2(t^2+6)^2\equiv A(t+3-\sqrt{3})^2(t+3+\sqrt{3})^3+B(t+3-\sqrt{3})(t+3+\sqrt{3})^3+C(t+3+\sqrt{3})^3$$

$$+D(t+3-\sqrt{3})^3(t+3+\sqrt{3})^2+E(t+3-\sqrt{3})^3(t+3+\sqrt{3})+F(t+3-\sqrt{3})^3$$

$$\begin{cases}A=\frac{7}{3}\sqrt{3}\\ B=-6+\sqrt{3}\\ C=12\sqrt{3}-18\\ D=-\frac{7}{3}\sqrt{3}\\ E=-6-\sqrt{3}\\ F=-12\sqrt{3}-18\end{cases}$$

$$=\frac{7}{3}\sqrt{3}\int\frac{1}{t+3-\sqrt{3}}\mathrm{d}t+(-6+\sqrt{3})\int\frac{1}{(t+3-\sqrt{3})^2}\mathrm{d}t+(12\sqrt{3}-18)\int\frac{1}{(t+3-\sqrt{3})^3}\mathrm{d}t$$

$$-\frac{7}{3}\sqrt{3}\int\frac{1}{t+3+\sqrt{3}}\mathrm{d}t+(-6-\sqrt{3})\int\frac{1}{(t+3+\sqrt{3})^2}\mathrm{d}t+(-12\sqrt{3}-18)\int\frac{1}{(t+3+\sqrt{3})^3}\mathrm{d}t$$

$$=(6-\sqrt{3})\frac{1}{t+3-\sqrt{3}}+(9-6\sqrt{3})\frac{1}{(t+3-\sqrt{3})^2}+(6+\sqrt{3})\frac{1}{t+3+\sqrt{3}}+$$

$$+(9+6\sqrt{3})\frac{1}{(t+3+\sqrt{3})^2}+\frac{7}{\sqrt{3}}\ln\left|\frac{t+3-\sqrt{3}}{t+3+\sqrt{3}}\right|+C=$$

$$=-\frac{1}{2}(\frac{\sqrt{3-2x^2}}{(x-1)^2}+6\frac{\sqrt{3-2x^2}}{x-1})+7\ln\left|\frac{\sqrt{3-2x^2}+2x-3}{x-1}\right|+C$$

17.120

$$\int \frac{\mathrm{d}x}{x^2\sqrt{1-4x+x^2}} = \left| \begin{array}{c} \sqrt{1-4x+x^2} = xt+1 \\ 1-4x+x^2 = x^2t^2+2xt+1 \\ -4x+x^2 = x^2t^2+2xt \\ -4+x = xt^2+2t \\ x-xt^2 = 2t+4 \\ x(1-t^2) = 2t+4 \\ x = \dfrac{2t+4}{1-t^2} \\ \mathrm{d}x = \dfrac{2(1-t^2)+2t(2t+4)}{(2t+4)^2}\mathrm{d}t \\ \mathrm{d}x = \dfrac{2t^2+8t+2}{(2t+4)^2}\mathrm{d}t \\ \sqrt{1-4x+x^2} = \dfrac{2t+4}{1-t^2}\cdot t+1 \\ \sqrt{1-4x+x^2} = \dfrac{t^2+4t+1}{1-t^2} \end{array} \right| =$$

$$= \int \frac{(1-t^2)^2}{(2t+4)^2}\cdot\frac{1-t^2}{t^2+4t+1}\cdot\frac{2t^2+8t+2}{(2t+4)^2}\mathrm{d}t$$

$$= -\frac{1}{2}\int\frac{t^2-1}{(t+2)^2}\mathrm{d}t$$

$$= -\frac{1}{2}\left(\int \mathrm{d}t - 4\int\frac{\mathrm{d}t}{t+2} + 3\int\frac{\mathrm{d}t}{(t+2)^2}\right)$$

$$= -\frac{1}{2}\left(t - 4\ln|t+2| - \frac{3}{t+2}\right) + C$$

$$= -\frac{\sqrt{1-4x+x^2}}{x} + 2\ln\left|\frac{\sqrt{1-4x+x^2}+2x-1}{x}\right| + C$$

17.121

$$\int \frac{\mathrm{d}x}{x^3\sqrt{1+x^2}} \rightarrow \mathbf{17.117}$$

17.122

$$\int \frac{\mathrm{d}x}{x^4\sqrt{3-2x+x^2}} = \sqrt{3}\int\frac{\mathrm{d}x}{x^4\sqrt{9-6x+3x^2}} =$$

$$\left|\begin{array}{c}\sqrt{9-6x+3x^2}=xt+3\\ 9-6x+3x^2=x^2t^2+6xt+9\\ -6x+3x^2=x^2t^2+6xt\\ -6+3x=xt^2+6t\\ 3x-xt^2=6t+6\\ x(3-t^2)=6t+6\\ x=\frac{6t+6}{3-t^2}\\ \mathrm{d}x=\frac{6(3-t^2)+2t(6t+6)}{(3-t^2)^2}\mathrm{d}t\\ \mathrm{d}x=\frac{6t^2+12t+18}{(3-t^2)^2}\mathrm{d}t\\ \sqrt{9-6x+3x^2}=\frac{6t+6}{3-t^2}\cdot t+3\\ \sqrt{9-6x+3x^2}=\frac{3t^2+6t+9}{3-t^2}\end{array}\right|$$

$$=\int\frac{(3-t^2)^4}{1296(t+1)^4}\cdot\frac{3-t^2}{3t^2+6t+9}\cdot\frac{6t^2+12t+18}{(3-t^2)^2}\mathrm{d}t=\frac{1}{648}\int\frac{(3-t^2)^3}{(1+t)^4}\mathrm{d}t$$

$$=\frac{1}{648}\left(\int(-t^2+4t-1)\mathrm{d}t-\int\frac{-16t^3-36t^2+28}{(t+1)^4}\mathrm{d}t\right)=$$

$$\left|-16t^3-36t^2+28=-16(t+1)^3+12(t+1)^2+24(t+1)+8\right|$$

$$=\frac{1}{648}\left(-\int t^2\mathrm{d}t+4\int t\mathrm{d}t-\int\mathrm{d}t+12\int\frac{\mathrm{d}t}{(t+1)^2}+24\int\frac{\mathrm{d}t}{(t+1)^3}+8\int\frac{\mathrm{d}t}{(1+t)^4}-16\int\frac{\mathrm{d}t}{t+1}\right)$$

$$=\frac{1}{648}\left(-\frac{1}{3}t^3+2t^2-t-\frac{12}{t+1}-\frac{12}{(t+1)^2}-\frac{8}{3}\cdot\frac{1}{(t+1)^3}-16\ln|t+1|\right)+C$$

$$=-\frac{1}{9}\cdot\frac{\sqrt{3-2x+x^2}}{x^3}-\frac{5}{54}\cdot\frac{\sqrt{3-2x+x^2}}{x^2}-\frac{1}{54}\cdot\frac{\sqrt{3-2x+x^2}}{x}-\frac{2\sqrt{3}}{81}\ln\left|\frac{\sqrt{9-6x+3x^2}+x-3}{x}\right|+C$$

17.123

$$\int\frac{\mathrm{d}x}{(x-2)^4\sqrt{1-4x+x^2}}$$

$$\left|\begin{array}{c} \sqrt{1-4x+x^2}=t-x \\ 1-4x+x^2=t^2-2tx+x^2 \\ 1-4x=t^2-2tx \\ t^2-1=2tx-4x \\ x(2t-4)=t^2-1 \\ x=\dfrac{t^2-1}{2t-4} \\ \mathrm{d}x=\dfrac{2t(2t-4)-2(t^2-1)}{(2t-4)^2}\mathrm{d}t \\ \mathrm{d}x=\dfrac{2t^2-8t+2}{(2t-4)^2}\mathrm{d}t \\ x-2=\dfrac{t^2-1-4t+8}{2t-4} \\ x-2=\dfrac{t^2-4t+7}{2t-4} \\ \sqrt{1-4x+x^2}=t-\dfrac{t^2-1}{2t-4} \\ \sqrt{1-4x+x^2}=\dfrac{t^2-4t+1}{2t-4} \end{array}\right|$$

$$= \int \frac{(2t-4)^4}{(t^2-4t+7)^4}\cdot\frac{2t-4}{t^2-4t+1}\cdot\frac{2t^2-8t+2}{(2t-4)^2}\mathrm{d}t$$

$$= 2\int \frac{(2t-4)^3}{(t^2-4t+7)^4}\mathrm{d}t$$

$$= 8\int \frac{(2t-4)(t^2-4t+7)-3(2t-4)}{(t^2-4t+7)^4}\mathrm{d}t$$

$$= \int \frac{8(2t-4)}{(t^2-4t+7)^3}\mathrm{d}t - \int \frac{24(2t-4)}{(t^2-4t+7)^4}\mathrm{d}t$$

$$= -\frac{4}{(t^2-4t+7)^2} + \frac{8}{(t^2-4t+7)^3} + C$$

$$= \frac{1}{27}\cdot\frac{(2x^2-8x+11)\sqrt{1-4x+x^2}}{(x-2)^3} + C$$

$$= \frac{2}{27}\cdot\frac{\sqrt{1-4x+x^2}}{x-2} + \frac{1}{9}\cdot\frac{\sqrt{1-4x+x^2}}{(x-2)^3} + C$$

18 Całki funkcji przestępnych.

18.1 § Całki funkcji trygonometrycznych.

18.30

$$\int \cos 5x \cos 7x dx = \int \cos 7x \cos 5x dx = \int \frac{1}{2}[\cos(7x+5x)+\cos(7x-5x)]dx =$$
$$= \int \frac{1}{2}(\cos 12x + \cos 2x)dx = \frac{1}{24}\sin 12x + \frac{1}{4}\sin 2x + C$$

18.31

$$\int \sin 3x \cos 2x dx = \int \frac{1}{2}[\sin(3x+2x)+\sin(3x-2x)]dx = \int \frac{1}{2}[\sin 5x + \sin x]dx =$$

$$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$$

18.32

$$\int \cos 2x \cos 3x dx = \int \frac{1}{2}[\cos(2x+3x) + \cos(2x-3x)]dx = \int \frac{1}{2}[\cos 5x + \cos(-x)]dx =$$
$$= \int \frac{1}{2}[\cos 5x + \cos x]dx = \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

18.33

$$\int \sin x \cos 3x dx = \int \frac{1}{2}[\sin(x+3x) + \sin(x-3x)]dx = \int \frac{1}{2}[\sin 4x + \sin(-2x)]dx =$$
$$= \int \frac{1}{2}[\sin 4x - \sin 2x]dx = -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$$

18.34

$$\int \cos 2x \sin 4x dx = \int \sin 4x \cos 2x dx = \int \frac{1}{2}[\sin(4x+2x) + \sin(4x-2x)]dx =$$
$$= \int \frac{1}{2}[\sin 6x + \sin 2x]dx = -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + C$$

18.35

$$\int \sin 2x \sin 5x dx = \int \frac{1}{2}[\cos(2x-5x) - \cos(2x+5x)]dx =$$
$$= \int \frac{1}{2}[\cos(-3x) - \cos 7x]dx = \int \frac{1}{2}[\cos 3x - \cos 7x]dx = \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$$

18.36

$$\int \cos x \cos 3x dx = \int \frac{1}{2}[\cos(x+3x) + \cos(x-3x)]dx = \int \frac{1}{2}[\cos 4x + \cos(-2x)]dx =$$
$$= \int \frac{1}{2}[\cos 4x + \cos 2x]dx = \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + C$$

18.37

$$\int \sin 3x \sin x dx = \int \frac{1}{2}[\cos(3x-x) - \cos(3x+x)]dx = \int \frac{1}{2}[\cos 2x - \cos 4x]dx =$$
$$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$$

18.38

$$\int \sin 5x \sin 2x dx = \int \frac{1}{2}[\cos(5x-2x) - \cos(5x+2x)]dx = \int \frac{1}{2}[\cos 3x - \cos 7x]dx =$$
$$= \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$$

18.39

$$\int \sin^3 x dx = \int (1-\cos^2 x)\sin x dx = \begin{vmatrix} t=\cos x \\ dt=-\sin x dx \end{vmatrix} = \int (t^2-1)dt =$$
$$= \frac{1}{3}t^3 - t + C = \frac{1}{3}\cos^3 x - \cos x + C$$

18.40

$$\int \sin^4 x dx = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C$$

Wzór redukcyjny

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

18.41

$$\int \cos^4 x dx = \int \sin^4(\frac{\pi}{2}+x)dx = \begin{vmatrix} u=\frac{\pi}{2}+x \\ du=dx \end{vmatrix} = \int \sin^4 u du =$$
$$= -\frac{1}{4}\sin^3 u\cos u - \frac{3}{8}\sin u\cos u + \frac{3}{8}u + C =$$
$$= -\frac{1}{4}\sin^3(\frac{\pi}{2}+x)\cos(\frac{\pi}{2}+x) - \frac{3}{8}\sin(\frac{\pi}{2}+x)\cos(\frac{\pi}{2}+x) + \frac{3}{8}(\frac{\pi}{2}+x) + C =$$
$$= \frac{1}{4}\sin^3 x\cos x + \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C$$

18.42

$$\int \cos^5 x dx = \int (1-\sin^2 x)^2\cos x dx = \begin{vmatrix} t=\sin x \\ dt=\cos x dx \end{vmatrix} = \int (1-t^2)^2 dt = \int (t^4-2t^2+1)dt =$$
$$= \frac{1}{5}t^5 - \frac{2}{3}t^3 + t + C = \frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x + C$$

18.43

$$\int \sin^5 x dx = \int (1-\cos^2 x)^2\sin x dx = \begin{vmatrix} t=\cos x \\ dt=-\sin x dx \end{vmatrix} = -\int (1-t^2)^2 dt =$$
$$= -\int (t^4-2t^2+1)dt = -\frac{1}{5}t^5 + \frac{2}{3}t^3 - t + C = -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$$

18.44

$$\int \tan^5 x dx = \begin{vmatrix} t=\tan x \\ \frac{dt}{t^2+1}=dx \end{vmatrix} = \int \frac{t^5}{t^2+1}dt = \int \frac{t^3(t^2+1)-t(t^2+1)+t}{t^2+1}dt =$$

$$= \int t^3 dt - \int t dt + \int \frac{\frac{1}{2}(t^2+1)'}{t^2+1} dt = \frac{1}{4}t^4 + \frac{1}{2}t^2 + \frac{1}{2}\ln|t^2+1| + C =$$
$$= \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + \frac{1}{2}\ln|\tan^2 x + 1| + C =$$
$$= \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x - \ln|\cos x| + C$$

18.45

$$\int \cot^4 x dx = \begin{vmatrix} t = \cot x \\ -\frac{dt}{t^2+1} = x \end{vmatrix} = -\int \frac{t^4}{t^2+1} dt = -\int \frac{(t^2-1)(t^2+1)+1}{t^2+1} dt =$$
$$= -\int (t^2-1)dt - \int \frac{dt}{t^2+1} = -\frac{1}{3}t^3 + t - \arctan(t) + C =$$
$$= -\frac{1}{3}\cot^3 x + \cot x - \arctan(\cot x) + C =$$
$$= -\frac{1}{3}\cot^3 x + \cot x - \arctan(\tan(\frac{\pi}{2} - x)) + C =$$
$$= -\frac{1}{3}\cot^3 x + \cot x + x + C$$

18.46

$$\int ctg^6 x dx = \begin{vmatrix} t = \cot x \\ -\frac{dt}{t^2+1} = x \end{vmatrix} = -\int \frac{t^6}{t^2+1} dt = -\int \frac{t^4(t^2+1) - t^2(t^2+1) + (t^2+1) - 1}{t^2+1} dt =$$
$$= -\int (t^4 - t^2 + 1)dt + \int \frac{dt}{t^2+1} = -\frac{1}{5}t^5 + \frac{1}{3}t^3 - t + \arctan(t) + C =$$
$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x + \arctan(\cot x) + C =$$
$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x + \arctan(\tan(\frac{\pi}{2} - x)) + C =$$
$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + C$$

18.47

$$\int \sin^3 x \cos^4 x dx = \int \sin x(1-\cos^2 x)\cos^4 x dx = \begin{vmatrix} t = \cos x \\ dt = -\sin x dx \end{vmatrix} =$$
$$= -\int (1-t^2)t^4 dt = \int (t^6 - t^4)dt = \frac{1}{7}t^7 - \frac{1}{5}t^5 + C = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

18.48

$$\int \sin^7 x \cos^6 x dx = \int \sin x(1-\cos^2 x)^3 \cos^6 x dx = \begin{vmatrix} t = \cos x \\ dt = -\sin x dx \end{vmatrix} =$$
$$= -\int (1-t^2)^3 t^6 dt = \int (t^{12} - 3t^{10} + 3t^8 - t^6)dt = \frac{1}{13}t^{13} - \frac{3}{11}t^{11} + \frac{1}{3}t^9 - \frac{1}{7}t^7 + C =$$
$$= \frac{1}{13}\cos^{13} x - \frac{3}{11}\cos^{11} x + \frac{1}{3}\cos^9 x - \frac{1}{7}\cos^7 x + C$$

18.49

$$\int \sin^5 x \cos^2 x dx = \int \sin x (1-cos^2 x)^2 \cos^2 x dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x dx \end{array} \right| =$$
$$= -\int (1-t^2)^2 t^2 dt = -\int (t^6 - 2t^4 + t^2) dt = -\frac{1}{7}t^7 + \frac{2}{5}t^5 - \frac{1}{3}t^3 + C =$$
$$= -\frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

18.50

$$\int \sin^2 x \cos^2 x dx = \int \frac{1}{4}\sin^2 2x dx = \int \frac{1-\cos 4x}{8} dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

18.51

$$\int \sin^3 x \cos^3 x dx = \int \sin^3 x \cos x (1-\sin^2 x) dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x \end{array} \right| =$$
$$= \int t^3 (1-t^2) dt = \int (t^3 - t^5) dt = \frac{1}{4}t^4 - \frac{1}{6}t^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

18.52

$$\int \sin^4 x \cos^5 x dx = \int \sin^4 x \cos x (1-\sin^2 x)^2 dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos dx \end{array} \right| =$$
$$= \int t^4 (1-t^2)^2 dt = \int (t^4 - 2t^6 + t^8) dt = \frac{1}{5}t^5 - \frac{2}{7}t^7 + \frac{1}{9}t^9 + C =$$
$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

18.53

$$\int \frac{\cos x dx}{\sin^8 x} = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int \frac{dt}{t^8} = -\frac{1}{7t^7} + C = -\frac{1}{7\sin^7 x} + C$$

18.54

$$\int \sin x \tan x dx = \int \frac{\sin^2 x}{\cos x} dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int \frac{t^2}{1-t^2} dt = \int \frac{t^2 - 1 + 1}{1-t^2} =$$
$$= -\int dt + \int \frac{dt}{1-t^2} = -t + \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C = -\sin x + \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C$$

18.55

$$\int \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int t^{-\frac{2}{3}} dt = 3\sqrt[3]{t} + C = 3\sqrt[3]{\sin x} + C$$

18.56

$$\int \frac{\sin x dx}{\sqrt[3]{1+2\cos x}} = \left| \begin{array}{l} t = 1+2\cos x \\ dt = -2\sin x dx \\ -\frac{1}{2}dt = \sin x dx \end{array} \right| = -\frac{1}{2}\int t^{-\frac{1}{3}}dt = -\frac{3}{4}t^{\frac{2}{3}} + C = -\frac{3}{4}\sqrt[3]{(1+2\cos x)^2} + C$$

18.57

$$\int \frac{\sin 2x dx}{\sqrt{1+\cos^2 x}} = \int \frac{2\sin x\cos x}{\sqrt{1+\cos^2 x}}dx = \int \frac{-(1+\cos^2 x)'}{\sqrt{1+\cos^2 x}}dx = -2\sqrt{1+\cos^2 x} + C$$

18.58

$$\int \frac{\sin 2x}{1+\sin^2 x}dx = \int \frac{2\sin x\cos x}{1+\sin^2 x}dx = \int \frac{(1+\sin^2 x)'}{1+\sin^2 x}dx = \ln|1+\sin^2 x| + C$$

18.59

$$\int \frac{\sin 2x dx}{\sqrt{1-sin^4 x}} = \int \frac{2\sin x\cos x dx}{\sqrt{1-sin^4 x}} = \left| \begin{array}{l} t = \sin^2 x \\ dt = 2\sin x\cos x dx \end{array} \right| = \int \frac{dt}{\sqrt{1-t^2}} =$$
$$= \arcsin(t) + C = \arcsin(\sin^2 x) + C$$

18.60

$$\int \frac{\cos^3 x}{\sin^2 x}dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^2 x}dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int \frac{1-t^2}{t^2}dt =$$
$$= \int \frac{dt}{t^2} - \int dt = -\frac{1}{t} - t + C = -\frac{1}{\sin x} - \sin x + C = -\frac{1+\sin^2 x}{\sin x} + C$$

18.61

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x - \sin x\cos x + \cos^2 x}dx = \int (\sin x + \cos x)dx = -\cos x + \sin x + C$$

18.62

$$\int \frac{dx}{\sin^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x}dx = \int \frac{dx}{\sin x} + \int \frac{\cos^2 x}{\sin^3 x}dx = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\int \frac{dx}{\sin x} =$$
$$= -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln|\tan\frac{x}{2}| + C$$

całki obliczone pomocniczo:

$$\int \frac{dx}{\sin x} = \left| \begin{array}{l} u = \tan\frac{x}{2} \\ \frac{2du}{1+u^2} = dx \\ \frac{2u}{1+u^2} = \sin x \end{array} \right| = \int \frac{du}{u} = \ln|u| + C = \ln|\tan\frac{x}{2}| + C$$
$$\int \frac{\cos^2 x}{\sin^3 x}dx = \int \frac{\cos x}{\sin^3 x}\cos x dx = \left| \begin{array}{ll} u = \cos x & dv = \frac{\cos x}{\sin^3 x}dx \\ du = -\sin x & v = -\frac{1}{2\sin^2 x} \end{array} \right| =$$

$$= -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\int \frac{dx}{\sin x}$$

$$\int \frac{\cos x}{\sin^3 x}dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x \end{array} \right| = \int \frac{dt}{t^3} = -\frac{1}{2t^2} + C = -\frac{1}{2\sin^2 x} + C$$

18.63

$$\int \frac{dx}{\cos^3 x} = \int \frac{dx}{\sin^3(x+\frac{\pi}{2})} = \left| \begin{array}{l} t = x + \frac{\pi}{2} \\ dt = dx \end{array} \right| = \int \frac{dt}{\sin^3 t} = ...$$

korzystając z rozwiązania w przykładzie (18.62) otrzymujemy:

$$... = -\frac{\cos t}{2\sin^2 t} + \frac{1}{2}\ln|\tan\frac{t}{2}| + C = -\frac{\cos(x+\frac{\pi}{2})}{2\sin^2(x+\frac{\pi}{2})} + \frac{1}{2}\ln|\tan\frac{(x+\frac{\pi}{2})}{2}| + C =$$

$$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln|\tan(\frac{x}{2}+\frac{\pi}{4})| + C$$

18.64

$$\int \frac{dx}{\sin^4 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x}dx = \int \frac{dx}{\sin^2 x} + \int \frac{\cos^2 x}{\sin^4 x}dx = -\cot x - \frac{1}{3}\cot^3 x + C$$

$$\int \frac{\cos^2 x}{\sin^4 x}dx = \int \frac{\cot^2 x}{\sin^2 x}dx = \left| \begin{array}{l} t = \cot x \\ -dt = \frac{dx}{\sin^2 x} \end{array} \right| = -\int t^2 dt = -\frac{1}{3}\cot^3 x + C$$

18.65

$$\int \frac{dx}{\cos^5 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^5 x}dx = \int \frac{\sin^2 x}{\cos^5 x} + \int \frac{dx}{\cos^3 x} = -\frac{\sin x}{3\cos^3 x} + \frac{3}{4}\int \frac{dx}{\cos^3 x} =$$

$$= \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \frac{3}{8}\ln|\tan(\frac{x}{2}+\frac{\pi}{4})| + C$$

$$\int \frac{\sin^2 x}{\cos^5 x}dx = \left| \begin{array}{ll} u = \sin x & dv = \frac{\sin x}{\cos^5 x}dx \\ du = \cos dx & v = \frac{1}{4\cos^4 x} \end{array} \right| = \frac{\sin x}{4\cos^4 x} - \frac{1}{4}\int \frac{dx}{\cos^3 x}$$

$$\int \frac{\sin x}{\cos^5 x}dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \end{array} \right| = -\int \frac{dt}{t^5} = \frac{1}{4t^4} + C = \frac{1}{4\cos^4 x} + C$$

18.66

$$\int \frac{dx}{\sin^7 x} = \left| \begin{array}{l} u = \tan\frac{x}{2} \\ \frac{2du}{1+u^2} = dx \\ \frac{2u}{1+u^2} = \sin x \end{array} \right| = \int \frac{\frac{2du}{1+u^2}}{(\frac{2u}{1+u^2})^7} = \int \frac{(u^2+1)^6}{64u^7}du =$$

$$= \frac{1}{64}\int \frac{u^{12} + 6u^{10} + 15u^8 + 20u^6 + 15u^4 + 6u^2 + 1}{u^7}du =$$

$$= \frac{1}{64}\int \left(u^5 + 6u^3 + 15u + \frac{20}{u} + \frac{15}{u^3} + \frac{6}{u^5} + \frac{1}{u^7}\right)du =$$

$$= \frac{1}{384}u^6 + \frac{3}{128}u^4 + \frac{15}{128}u^2 + \frac{5}{16}\ln|u| - \frac{15}{128u^2} - \frac{3}{128u^4} - \frac{1}{384u^6} + C =$$

$$= \frac{1}{384}\tan^6\frac{x}{2} + \frac{3}{128}\tan^4\frac{x}{2} + \frac{15}{128}\tan^2\frac{x}{2} + \frac{5}{16}\ln|\tan\frac{x}{2}| - \frac{15}{128\tan^2\frac{x}{2}} - \frac{3}{128\tan^4\frac{x}{2}} - \frac{1}{384\tan^6\frac{x}{2}} + C$$

18.67

$$\int \frac{dx}{\sin^3 x \cos x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos x} dx = \int \frac{dx}{\sin x \cos x} + \int \frac{\cos x}{\sin^3 x} dx =$$
$$= \int \frac{dx}{\frac{1}{2}\sin 2x} + \int \frac{\cot x}{\sin^2 x} dx = \ln|\tan x| - \frac{1}{2}\cot^2 x + C$$
$$\int \frac{dx}{\frac{1}{2}\sin 2x} = \begin{vmatrix} t = 2x \\ \frac{1}{2}dt = dx \end{vmatrix} = \int \frac{dt}{\sin t} = \ln|\tan \frac{t}{2}| + C = \ln|\tan x| + C$$
$$\int \frac{\cot x}{\sin^2 x} dx = \begin{vmatrix} t = \cot x \\ -dt = \frac{dx}{\sin^2 x} \end{vmatrix} = -\int t dt = -\frac{1}{2}\cot^2 x + C$$

18.68

$$\int \frac{dx}{\sin x \cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} dx = \int \frac{\sin x}{\cos^3 x} dx + \int \frac{dx}{\sin x \cos x} =$$
$$= \int \frac{\tan x}{\cos^2 x} dx + \int \frac{dx}{\frac{1}{2}\sin 2x} = \frac{1}{2}\tan^2 x + \ln|\tan x| + C$$
$$\int \frac{dx}{\frac{1}{2}\sin 2x} = \begin{vmatrix} t = 2x \\ \frac{1}{2}dt = dx \end{vmatrix} = \int \frac{dt}{\sin t} = \ln|\tan \frac{t}{2}| + C = \ln|\tan x| + C$$
$$\int \frac{\tan x}{\cos^2 x} dx = \begin{vmatrix} t = \tan x \\ dt = \frac{dx}{\cos^2 x} \end{vmatrix} = \int t dt = \frac{1}{2}\tan^2 x + C$$

18.69

$$\int \frac{dx}{\sin^5 x \cos^3 x} == \int \frac{\sin^2 x + \cos^2 x}{\sin^5 x \cos^3 x} dx =$$
$$= \int \frac{dx}{\sin^3 x \cos^3 x} + \int \frac{dx}{\sin^5 x \cos x} =$$
$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos^3 x} + \int \frac{\sin^2 x + \cos^2 x}{\sin^5 x \cos x} dx =$$
$$= \int \frac{dx}{\sin x \cos^3 x} + \int \frac{dx}{\sin^3 \cos x} + \int \frac{dx}{\sin^3 x \cos x} + \int \frac{\cos x}{\sin^5 x} dx =$$
$$= \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} dx + 2\int \frac{\sin^2 x + \cos^2 x}{\sin^3 \cos x} + \int \frac{\cos x}{\sin^5 x} dx =$$
$$= \int \frac{\sin x}{\cos^3 x} dx + \int \frac{dx}{\sin x \cos x} + 2\int \frac{dx}{\sin x \cos x} + 2\int \frac{\cos x}{\sin^3 x} + \int \frac{\cos x}{\sin^5 x} dx =$$
$$= \int \frac{\sin x}{\cos^3 x} dx + 3\int \frac{dx}{\sin x \cos x} + 2\int \frac{\cos x}{\sin^3 x} dx + \int \frac{\cos x}{\sin^5 x} dx =$$
$$= \frac{1}{2\cos^2 x} + 3\ln|\tan x| - \frac{1}{\sin^2 x} - \frac{1}{4\sin^4 x} + C$$

całki obliczone pomocniczo

$$\int \frac{\sin x}{\cos^3 x} dx = \begin{vmatrix} t = \cos x \\ -dt = \sin x dx \end{vmatrix} = -\int \frac{dt}{t^3} = \frac{1}{2t^2} + C = \frac{1}{2\cos^2 x} + C$$
$$\int \frac{dx}{\sin x \cos x} = \int \frac{dx}{\frac{1}{2}\sin 2x} = \begin{vmatrix} t = 2x \\ \frac{1}{2}dt = dx \end{vmatrix} = \int \frac{dt}{\sin t} = \ln|\tan \frac{t}{2}| + C = \ln|\tan x| + C$$
$$\int \frac{\cos x}{\sin^3 x} dx = \begin{vmatrix} t = \sin x \\ dt = \cos x \end{vmatrix} = \int \frac{dt}{t^3} = -\frac{1}{2t^2} + C = -\frac{1}{2\sin^2 x} + C$$

$$\int \frac{\cos x}{\sin^5 x}dx = \begin{vmatrix} t = \sin x \\ dt = \cos x \end{vmatrix} = \int \frac{dt}{t^5} = -\frac{1}{4t^4} + C = -\frac{1}{4\sin^4 x} + C$$

18.70

$$\int \frac{dx}{\sin^2 x \cos^4 x} =$$
$$= \int \frac{\sin^2 + \cos^2 x}{\sin^2 x \cos^4 x}dx =$$
$$= \int \frac{dx}{\cos^4 x} + \int \frac{dx}{\sin^2 x \cos^2 x} =$$
$$= \int \frac{\sin^2 + \cos^2 x}{\cos^4 x}dx + \int \frac{\sin^2 + \cos^2 x}{\sin^2 x \cos^2 x}dx =$$
$$= \int \frac{\sin^2}{\cos^4 x}dx + \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} =$$
$$= \int \frac{\tan^2 x}{\cos^2 x}dx + 3\int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} =$$
$$= \frac{1}{3}\tan^3 x + 3\tan x - \cot x + C$$
$$\int \frac{\tan^2 x}{\cos^2 x}dx = \begin{vmatrix} t = \tan x \\ dt = \frac{dx}{\cos^2 x} \end{vmatrix} = \int t^2 dt = \frac{1}{3}t^3 + C = \frac{1}{3}\tan^3 x + C$$

18.71

$$\int \frac{\sin^4 x}{\cos^3 x}dx = \int \frac{(1-\cos^2 x)\sin^2 x}{\cos^3 x}dx = \int \frac{\sin^2 x}{\cos^3 x}dx - \int \frac{\sin^2 x}{\cos x}dx =$$
$$= \int \frac{1-\cos^2 x}{\cos^3 x}dx - \int \frac{1-\cos^2 x}{\cos x}dx = \int \frac{dx}{\cos^3 x} - \int \frac{dx}{\cos x} - \int \frac{dx}{\cos x} + \int \cos x dx =$$
$$= \int \frac{dx}{\cos^3 x} - 2\int \frac{dx}{\cos x} + \sin x =$$
$$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln|\tan(\frac{x}{2} + \frac{\pi}{4})| - 2\ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + \sin x + C =$$
$$= \frac{\sin x}{2\cos^2 x} - \frac{3}{2}\ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + \sin x + C$$

w przykładzie wykorzystano rozwiązania całek z przykładów (18.62), (18.63)

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(x + \frac{\pi}{2})} = \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + C$$

18.72

$$\int \frac{\sin^4 x dx}{\cos x} = \int \frac{(1-\cos^2 x)}{\cos x}dx = \int \frac{\cos^4 x - 2\cos^2 x + 1}{\cos x}dx =$$
$$= \int \cos^3 x dx - 2\int \cos x + \int \frac{dx}{\cos x} =$$
$$= \int (1-\sin^2 x)\cos x dx - 2\sin x + \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| =$$

$$= -\sin x - \frac{1}{3}\sin^3 x + \ln|\tan(\frac{x}{2}+\frac{\pi}{4})| + C$$

$$\int (1-\sin^2 x)\cos x dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int (1-t^2)dt = t - \frac{1}{3}t^3 + C =$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

18.73

$$\int \frac{\cos^5 x dx}{\sin^3 x} = \int \frac{(1-\sin^2 x)^2 \cos x}{\sin^3 x} dx = \int \frac{\cos x}{\sin^3 x} dx - 2\int \frac{\cos x}{\sin x} dx + \int \sin x \cos x dx =$$

$$= \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int \frac{dt}{t^3} - 2\int \frac{dt}{t} + \int t dt = -\frac{1}{2t^2} - 2\ln|t| + \frac{1}{2}t^2 + C =$$

$$= -\frac{1}{2\sin^2 x} - 2\ln|\sin x| + \frac{1}{2}\sin^2 x + C$$

18.74

$$\int \frac{\sin^3 x dx}{\cos^8 x} = \int \frac{(1-\cos^2 x)\sin x}{\cos^8 x} dx = \int \frac{\sin x}{\cos^8 x} dx - \int \frac{\sin x}{\cos^6 x} dx = \left| \begin{array}{l} t = \cos x \\ -dt = \sin x dx \end{array} \right| =$$

$$= -\int \frac{dt}{t^8} + \int \frac{dt}{t^6} = \frac{1}{7t^7} - \frac{1}{5t^5} + C = \frac{1}{7\cos^7 x} - \frac{1}{5\cos^5 x} + C$$

18.75

$$\int \frac{\cos 2x dx}{\cos^3 x} = \int \frac{2\cos^2 x - 1}{\cos^3 x} dx = 2\int \frac{dx}{\cos x} - \int \frac{dx}{\cos^3 x} =$$

$$= \frac{3}{2}\ln|\tan(\frac{x}{2}+\frac{\pi}{4})| - \frac{\sin x}{2\cos^2 x} + C$$

w przykładzie wykorzystano całki z przykładów (18.71), (18.63)

18.76

$$\int \frac{dx}{5+4\cos x} = \left| \begin{array}{l} u = \tan\frac{x}{2} \\ \frac{2du}{1+u^2} = dx \\ \frac{1-u^2}{1+u^2} = \cos x \end{array} \right| = \int \frac{\frac{2du}{1+u^2}}{5+4\cdot\frac{1-u^2}{1+u^2}} = \int \frac{\frac{2du}{1+u^2}}{\frac{5+5u^2+4-4u^2}{1+u^2}} = \int \frac{2du}{9+u^2} =$$

$$= \frac{2}{9}\int \frac{du}{(\frac{u}{3})^2+1} = \left| \begin{array}{l} t = \frac{u}{3} \\ 3dt = du \end{array} \right| = \frac{2}{3}\int \frac{dt}{t^2+1} = \frac{2}{3}\arctan\left(\frac{1}{3}\tan\frac{x}{2}\right) + C$$

18.77

$$\int \frac{dx}{1+\sin x} = \left| \begin{array}{l} u = \tan\frac{x}{2} \\ \frac{2du}{1+u^2} = dx \\ \frac{2u}{1+u^2} = \sin x \end{array} \right| = \int \frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2}} = \int \frac{2du}{(u+1)^2} =$$

$$= -\frac{2}{u+1} + C = -\frac{2}{\tan\frac{x}{2}+1} + C$$

18.78

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2}\sin(x+\frac{\pi}{4})} = \frac{1}{\sqrt{2}} \ln|\tan(\frac{x}{2}+\frac{\pi}{8})| + C$$

18.79

$$\int \frac{\sin x \cos x dx}{\sin^4 x + \cos^4 x} = \ldots$$

zakładając, że $\cos x \neq 0$

$$\ldots = \int \frac{\frac{\sin x}{\cos^3 x}}{\frac{\sin^4 x}{\cos^4 x}+1} dx = \left| \begin{array}{c} t = \tan^2 x \\ \frac{1}{2}dt = \frac{\sin x}{\cos^3 x} dx \end{array} \right| = \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\arctan(t) + C =$$

$$= \frac{1}{2}\arctan(\tan^2 x) + C$$

18.80

$$\int \frac{3+\sin^2 x}{2\cos^2 x - \cos^4 x} dx = \left| \begin{array}{l} t = \tan x \\ \frac{dt}{t^2+1} = dx \\ \frac{t^2}{t^2+1} = \sin^2 x \\ \frac{1}{t^2+1} = \cos^2 x \end{array} \right| = \int \frac{3+\frac{t^2}{t^2+1}}{\frac{2}{t^2+1} - \frac{1}{(t^2+1)^2}} \cdot \frac{dt}{t^2+1} = \int \frac{\frac{4t^2+3}{t^2+1}}{\frac{2t^2+1}{t^2+1}} dt =$$

$$= \int \frac{4t^2+3}{2t^2+1} dt = \int \frac{2(2t^2+1)+1}{2t^2+1} dt = \int 2dt + \int \frac{dt}{2t^2+1} = 2t + \frac{1}{\sqrt{2}}\arctan(\sqrt{2}t) + C =$$

$$= 2\tan x + \arctan(\sqrt{2}\tan x) + C$$

$$\int \frac{dt}{2t^2+1} = \left| \begin{array}{c} u = \sqrt{2}t \\ \frac{du}{\sqrt{2}} = dt \end{array} \right| = \frac{1}{\sqrt{2}}\int \frac{du}{u^2+1} = \frac{1}{\sqrt{2}}\arctan(u) + C = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}t) + C$$

18.81

$$\int \frac{\cos x + \sin x}{(\sin x - \cos x)^2} dx = \left| \begin{array}{l} t = \sin x - \cos x \\ dt = (\cos x + \sin x)dx \end{array} \right| = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{\sin x - \cos x} + C$$

18.82

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx = \int \frac{-\cos 2x}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} dx = -\int \frac{\cos 2x}{1 - \frac{1}{2}\sin^2 2x} dx =$$

$$= \left| \begin{array}{l} t = \sin 2x \\ \frac{1}{2}dt = \cos 2x dx \end{array} \right| = -\frac{1}{2}\int \frac{dt}{1-\frac{1}{2}t^2} = \left| \begin{array}{c} u = \frac{1}{\sqrt{2}}t \\ \sqrt{2}du = dt \end{array} \right| = -\frac{1}{\sqrt{2}}\int \frac{du}{1-u^2} =$$

$$= -\frac{1}{2\sqrt{2}}\ln\left|\frac{1+u}{1-u}\right| + C = -\frac{1}{2\sqrt{2}}\ln\left|\frac{1+\frac{\sin 2x}{\sqrt{2}}}{1-\frac{\sin 2x}{\sqrt{2}}}\right| + C = -\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right| + C$$

18.83

$$\int \frac{\sin x\cos x}{1+\sin^4 x}dx = \left| \begin{array}{l} t=\sin^2 x \\ \frac{1}{2}dt = \sin x\cos x \end{array} \right| = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\arctan(t)+C = \frac{1}{2}\arctan(\sin^2 x)+C$$

18.84

$$\int \frac{dx}{(\sin^2 x+3\cos^2 x)^2} = \left| \begin{array}{l} t=\tan x \\ \frac{dt}{t^2+1}=dx \\ \frac{t^2}{t^2+1}=\sin^2 x \\ \frac{1}{t^2+1}=\cos^2 x \end{array} \right| = \int \frac{\frac{dt}{t^2+1}}{(\frac{t^2}{t^2+1}+\frac{3}{t^2+1})^2} = \int \frac{\frac{dt}{t^2+1}}{\frac{(t^2+3)^2}{(t^2+1)^2}} =$$

$$= \int \frac{t^2+1}{(t^2+3)^2}dt = \int \frac{t^2+3-2}{(t^2+3)^2}dt = \int \frac{dt}{t^2+3} - 2\int \frac{dt}{(t^2+3)^2} =$$

$$= \frac{1}{3}\int \frac{dt}{\frac{t^2}{3}+1} - \frac{2}{9}\int \frac{dt}{(\frac{t^2}{3}+1)^2} = \left| \begin{array}{l} u=\frac{t}{\sqrt{3}} \\ \sqrt{3}du = dt \end{array} \right| = \frac{1}{\sqrt{3}}\int \frac{du}{u^2+1} - \frac{2}{3\sqrt{3}}\int \frac{du}{(u^2+1)^2} = \ldots$$

korzystając z całki obliczonej w przykładzie (16.69) otrzymujemy:

$$\ldots = \frac{1}{\sqrt{3}}\arctan(u) - \frac{1}{3\sqrt{3}}\arctan(u) - \frac{1}{3\sqrt{3}}\cdot\frac{u}{u^2+1}+C=$$

$$= \frac{2}{3\sqrt{3}}\arctan\left(\frac{\tan x}{\sqrt{3}}\right) - \frac{1}{3\sqrt{3}}\cdot\frac{\frac{\tan x}{\sqrt{3}}}{\frac{\tan^2 x}{3}+1}+C$$

$$= \frac{2}{3\sqrt{3}}\arctan\left(\frac{\tan x}{\sqrt{3}}\right) - \frac{\tan x}{3\tan^2 x+9}+C$$

18.85

$$\int \frac{\sin^2 x\cos^2 x}{\sin^8 x+\cos^8 x}dx = \int \frac{\frac{1}{4}\sin^2 2x}{\left(\sin^4 x+\cos^4 x\right)^2 - \frac{1}{8}\sin^4 2x}dx = \int \frac{\frac{1}{4}\sin^2 2x}{\frac{1}{4}(1+\cos^2 2x)^2 - \frac{1}{8}\sin^4 2x}dx =$$

$$= \int \frac{\frac{1}{4}\sin^2 2x}{1-\sin^2 2x+\frac{1}{8}\sin^4 2x}dx = \int \frac{\frac{1}{8}\cdot\frac{2}{\cos^2 2x}}{\frac{1}{\sin^2 2x\cos^2 2x}-\frac{1}{\cos^2 2x}+\frac{1}{8}\tan^2 2x}dx = \int \frac{\frac{1}{8}\cdot\frac{2}{\cos^2 2x}dx}{1+\frac{1}{\tan^2 2x}+\frac{1}{8}\tan^2 2x}dx =$$

$$= \left| \begin{array}{l} t=\tan 2x \\ dt = \frac{2dx}{\cos^2 2x} \end{array} \right| = \int \frac{\frac{1}{8}dt}{1+\frac{1}{t^2}+\frac{1}{8}t^2} = \int \frac{t^2dt}{t^4+8t^2+8} = \ldots$$

$$t^4+8t^2+8 \equiv (t^2+a)(t^2+b)$$
$$t^4+8t^2+8 \equiv t^4+(a+b)t^2+ab$$
$$\begin{cases} a+b=8 \\ ab=8 \end{cases}$$
$$\begin{cases} a=4-2\sqrt{2} \\ b=4+2\sqrt{2} \end{cases} \vee \begin{cases} a=4+2\sqrt{2} \\ b=4-2\sqrt{2} \end{cases}$$

rozkład na ułamki proste:

$$\frac{t^2}{(t^2+4-2\sqrt{2})(t^2+4+2\sqrt{2})} \equiv \frac{At+B}{t^2+4-2\sqrt{2}} + \frac{Ct+D}{t^2+4+2\sqrt{2}}$$

$t^2 \equiv (At+B)(t^2+4-2\sqrt{2})+(Ct+D)(t^2+4+2\sqrt{2})$

$t^2 \equiv (A+C)t^3+(B+D)t^2+[(4+2\sqrt{2})A+(4-2\sqrt{2})C]t+(4+2\sqrt{2})B+(4-2\sqrt{2})D$

$$\begin{cases} A+C=0 \\ B+D=1 \\ (4+2\sqrt{2})A+(4-2\sqrt{2})C=0 \\ (4+2\sqrt{2})B+(4-2\sqrt{2})D=0 \end{cases}$$

$$\begin{cases} A=0 \\ B=\frac{1}{2}(1-\sqrt{2}) \\ C=0 \\ D=\frac{1}{2}(1+\sqrt{2}) \end{cases}$$

$$... = \int \frac{\frac{1}{2}(1-\sqrt{2})}{t^2+4-2\sqrt{2}}dt + \int \frac{\frac{1}{2}(1+\sqrt{2})}{t^2+4+2\sqrt{2}}dt =$$

$$= \frac{\frac{1}{2}(1-\sqrt{2})}{4-2\sqrt{2}} \int \frac{dt}{(\frac{t}{\sqrt{4-2\sqrt{2}}})^2+1} + \frac{\frac{1}{2}(1+\sqrt{2})}{4+2\sqrt{2}} \int \frac{dt}{(\frac{t}{\sqrt{4+2\sqrt{2}}})^2+1} =$$

$$= \left| \begin{array}{l} u=\frac{t}{\sqrt{4-2\sqrt{2}}} \\ \sqrt{4-2\sqrt{2}}du=dt \end{array} \right| + \left| \begin{array}{l} v=\frac{t}{\sqrt{4+2\sqrt{2}}} \\ \sqrt{4+2\sqrt{2}}dv=dt \end{array} \right| =$$

$$= \frac{\frac{1}{2}(1-\sqrt{2})}{\sqrt{4-2\sqrt{2}}} \int \frac{du}{u^2+1} + \frac{\frac{1}{2}(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \int \frac{dv}{v^2+1} =$$

$$= \frac{\frac{1}{2}(1-\sqrt{2})}{\sqrt{4-2\sqrt{2}}} \arctan u + \frac{\frac{1}{2}(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \arctan v + C =$$

$$= \frac{\frac{1}{2}(1-\sqrt{2})}{\sqrt{4-2\sqrt{2}}} \arctan\left(\frac{\tan 2x}{\sqrt{4-2\sqrt{2}}}\right) + \frac{\frac{1}{2}(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \arctan\left(\frac{\tan 2x}{\sqrt{4+2\sqrt{2}}}\right) + C$$

18.86

$$\int \frac{dx}{\sin^4 x+\cos^4 x} = ...$$

korzystając ze wzorów:

$$\sin^4 x = \frac{\cos(4x)-4\cos(2x)+3}{8} \qquad \cos^4 x = \frac{\cos(4x)+4\cos(2x)+3}{8}$$

otrzymujemy:

$$... = \int \frac{4dx}{\cos 4x+3} = \left| \begin{array}{l} u=4x \\ \frac{1}{4}du=dx \end{array} \right| = \int \frac{du}{\cos u+3} = \left| \begin{array}{l} t=\tan\frac{u}{2} \\ \frac{2dt}{t^2+1}=du \\ \frac{-t^2+1}{t^2+1}=\cos u \end{array} \right| = \int \frac{\frac{2dt}{t^2+1}}{\frac{-t^2+1}{t^2+1}+3} =$$

$$= \int \frac{\frac{2dt}{t^2+1}}{\frac{2t^2+4}{t^2+1}} = \int \frac{dt}{t^2+2} = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}} \arctan\left(\frac{t}{\sqrt{2}}\right) + C =$$

$$= \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan 2x}{\sqrt{2}}\right) + C$$

18.87

$$\int \frac{dx}{1-\sin^4 x} = \int \frac{dx}{(1-\sin^2 x)(1+\sin^2 x)} = \int \frac{dx}{\cos^2 x(1+\sin^2 x)} = \left| \begin{array}{l} t = \tan x \\ dt = \frac{dx}{\cos^2 x} \\ \frac{t^2}{t^2+1} = \sin^2 x \end{array} \right| =$$

$$= \int \frac{dt}{1+\frac{t^2}{t^2+1}} = \int \frac{t^2+1}{2t^2+1} dt = \int \frac{1}{2} dt + \frac{1}{2} \int \frac{dt}{2t^2+1} =$$

$$= \frac{1}{2} t - \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}t) + C = \frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x) + C$$

$$\int \frac{dt}{2t^2+1} = \left| \begin{array}{l} u = \sqrt{2}t \\ \frac{du}{\sqrt{2}} = dt \end{array} \right| = \frac{1}{\sqrt{2}} \int \frac{du}{u^2+1} = \frac{1}{\sqrt{2}} \arctan(u) + C$$

18.2 § Całki funkcji cyklometrycznych (kołowych).

18.91

$$\int \frac{x^2}{\sqrt{1-x^2}} \arcsin x dx = \left| \begin{array}{ll} u = \arcsin x & dv = \frac{x^2}{\sqrt{1-x^2}} dx \\ du = \frac{dx}{\sqrt{1-x^2}} & v = \frac{1}{2} \arcsin x - \frac{1}{2} x\sqrt{1-x^2} \end{array} \right| =$$

$$= \frac{1}{2} \arcsin^2 x - \frac{1}{2} x\sqrt{1-x^2} \arcsin x - \frac{1}{2} \int \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \int x dx =$$

$$= \frac{1}{4} \arcsin^2 x - \frac{1}{2} x\sqrt{1-x^2} \arcsin x + \frac{1}{4} x^2 + C$$

całki obliczone pomocniczo:

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \int \frac{x^2-1+1}{\sqrt{1-x^2}} dx = -\int \sqrt{1-x^2} dx + \int \frac{dx}{\sqrt{1-x^2}} =$$

$$= -\frac{1}{2} \arcsin x - \frac{1}{2} x\sqrt{1-x^2} + \arcsin x + C = \frac{1}{2} \arcsin x - \frac{1}{2} x\sqrt{1-x^2} + C$$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} = \left| \begin{array}{l} t = \arcsin \\ dt = \frac{dx}{\sqrt{1-x^2}} \end{array} \right| = \int t dt = \frac{1}{2} \arcsin^2 x + C$$

18.92

$$\int \frac{\arcsin x}{\sqrt{(1-x^2)^3}} dx = \left| \begin{array}{ll} u = \arcsin x & dv = \frac{dx}{\sqrt{(1-x^2)^3}} \\ du = \frac{dx}{\sqrt{1-x^2}} & v = \frac{x}{\sqrt{1-x^2}} \end{array} \right| = \frac{x \arcsin x}{\sqrt{1-x^2}} - \int \frac{x}{1-x^2} dx =$$

$$= \frac{x \arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln |1-x^2| + C$$

$$\int \frac{dx}{\sqrt{(1-x^2)^3}} = \left| \begin{array}{l} t = \arcsin x \\ dt = \frac{dx}{\sqrt{1-x^2}} \\ \sin t = x \end{array} \right| = \int \frac{dt}{1-\sin^2 t} = \int \frac{dt}{\cos^2 t} = \tan t + C =$$

$$= \tan(\arcsin x) + C = \tan(\arctan \frac{x}{\sqrt{1-x^2}}) + C = \frac{x}{\sqrt{1-x^2}} + C$$

$$\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}}$$

18.93

$$\int \frac{x^2}{1+x^2}\arctan x dx = \int \frac{x^2+1-1}{x^2+1}\arctan x dx = \int \arctan x - \int \frac{\arctan x}{x^2+1}dx =$$
$$= x\arctan x - \frac{1}{2}\ln|x^2+1| - \frac{1}{2}\arctan^2 x + C$$

całki obliczone pomocniczo:

$$\int \arctan x dx = \left| \begin{array}{ll} u=\arctan x & dv=dx \\ du=\frac{dx}{x^2+1} & v=x \end{array} \right| = x\arctan x - \int \frac{x}{x^2+1}dx =$$
$$= x\arctan x - \frac{1}{2}\ln|x^2+1| + C$$
$$\int \frac{\arctan x}{x^2+1}dx = \left| \begin{array}{l} t=\arctan \\ dt=\frac{dx}{x^2+1} \end{array} \right| = \int t dt = \frac{1}{2}t^2 + C = \frac{1}{2}\arctan^2 x + C$$

18.94

$$\int \frac{dx}{(1+9x^2)\sqrt{\arctan 3x}} = \left| \begin{array}{l} t=\arctan 3x \\ \frac{1}{3}dt=\frac{dx}{1+9x^2} \end{array} \right| = \frac{1}{3}\int \frac{dt}{\sqrt{t}} = \frac{2}{3}\sqrt{t}+C = \frac{2}{3}\sqrt{\arctan 3x}+C$$

18.95

$$\int \frac{dx}{(1+4x^2)(\arctan 2x)^2} = \left| \begin{array}{l} t=\arctan 2x \\ \frac{1}{2}dt=\frac{dx}{1+4x^2} \end{array} \right| = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t}+C = -\frac{1}{2\arctan 2x}+C$$

18.96

$$\int \frac{(\arctan x)^2}{x^2+1}dx = \left| \begin{array}{l} t=\arctan x \\ dt=\frac{dx}{x^2+1} \end{array} \right| = \int t^2 dt = \frac{1}{3}t^3+C = \frac{1}{3}\arctan^3 x + C$$

18.97

$$\int \frac{dx}{\sqrt{1-x^2}\arccos^2 x} = \left| \begin{array}{l} t=\arccos x \\ -dt=\frac{dx}{\sqrt{1-x^2}} \end{array} \right| = -\int \frac{dt}{t^2} = \frac{1}{t}+C = \frac{1}{\arccos x}+C$$

18.98

$$\int \frac{dx}{\sqrt{1-x^2}\arcsin x} = \left| \begin{array}{l} t=\arcsin x \\ dt=\frac{dx}{\sqrt{1-x^2}} \end{array} \right| = \int \frac{dt}{t} = \ln|t|+C = \ln|\arcsin x|+C$$

18.99

$$\int \frac{x\arctan x dx}{(1+x^2)^2} = \left| \begin{array}{ll} u=\arctan x & dv=\frac{x}{(x^2+1)^2}dx \\ du=\frac{dx}{x^2+1} & v=-\frac{1}{2(x^2+1)} \end{array} \right| = -\frac{\arctan x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{(x^2+1)^2} =$$
$$= -\frac{\arctan x}{2(x^2+1)} + \frac{1}{4}\arctan x + \frac{x}{4(x^2+1)} + C$$

18.100

$$\int \frac{x\arcsin x dx}{(1-x^2)^{\frac{3}{2}}} = \left| \begin{array}{ll} u=\arcsin x & dv=\frac{x}{(1-x^2)^{\frac{3}{2}}} \\ du=\frac{dx}{\sqrt{1-x^2}} & v=\frac{1}{\sqrt{1-x^2}} \end{array} \right| = \frac{\arcsin x}{\sqrt{1-x^2}} - \int \frac{dx}{1-x^2} =$$
$$= \frac{\arcsin x}{\sqrt{1-x^2}} - \ln\left|\frac{1+x}{1-x}\right| + C$$

18.101

$$\int x\arcsin x dx = \left| \begin{array}{ll} u=\arcsin x & dv=xdx \\ du=\frac{dx}{\sqrt{1-x^2}} & v=\frac{1}{2}x^2 \end{array} \right| = \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx =$$
$$= \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int \frac{x^2-1+1}{\sqrt{1-x^2}}dx = \frac{1}{2}x^2\arcsin x + \frac{1}{2}\int \sqrt{1-x^2}dx - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}} =$$
$$= \frac{1}{2}x^2\arcsin x + \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} - \frac{1}{2}\arcsin x + C =$$
$$= \frac{1}{2}(x^2 - \frac{1}{2})\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C$$

18.102

$$\int \frac{x\arctan x dx}{(x^2-1)^2} = \left| \begin{array}{ll} u=\arctan x & dv=\frac{x}{(x^2-1)^2}dx \\ du=\frac{dx}{x^2+1} & v=-\frac{1}{2(x^2-1)} \end{array} \right| = -\frac{\arctan x}{2(x^2-1)} + \frac{1}{2}\int \frac{dx}{(x^2+1)(x^2-1)} =$$
$$= -\frac{\arctan x}{2(x^2-1)} - \frac{1}{4}\arctan x - \frac{1}{8}\ln|x+1| + \frac{1}{8}\ln|x-1| + C$$
$$\int \frac{dx}{(x^2+1)(x^2-1)}$$

rozkład na ułamki proste:

$$\frac{1}{(x^2+1)(x^2-1)} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$
$$1 \equiv (Ax+B)(x^2-1) + C(x-1)(x^2+1) + D(x+1)(x^2+1)$$
$$1 \equiv (A+C+D)x^3 + (B-C+D)x^2 + (-A+C+D)x + (-A-C+D)$$
$$\begin{cases} A+C+D=0 \\ B-C+D=0 \\ -A+C+D=0 \\ -B-C+D=1 \end{cases}$$
$$\begin{cases} A=0 \\ B=-\frac{1}{2} \\ C=-\frac{1}{4} \\ D=\frac{1}{4} \end{cases}$$
$$\ldots = \int \frac{-\frac{1}{2}}{x^2+1}dx + \int \frac{-\frac{1}{4}}{x+1}dx + \int \frac{\frac{1}{4}}{x-1}dx =$$
$$= -\frac{1}{2}\arctan x - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C$$

18.103

$$\int x^2 \arctan x dx = \left| \begin{array}{ll} u = \arctan x & dv = x^2 dx \\ du = \frac{dx}{x^2+1} & v = \frac{1}{3}x^3 \end{array} \right| = \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{x^2+1}dx =$$
$$= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x(x^2+1)-x}{x^2+1}dx = \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln|x^2+1| + C$$

18.104

$$\int \frac{\arctan e^{\frac{1}{2}x}}{e^{\frac{1}{2}x}(1+e^x)}dx = \left| \begin{array}{l} t = e^{\frac{1}{2}x} \\ \ln t = \frac{1}{2}x \\ 2\ln t = x \\ \frac{2dt}{t} = dx \end{array} \right| = \int \frac{2\arctan t}{t^2(t^2+1)}dt = \left| \begin{array}{ll} u = \arctan t & dv = \frac{2dt}{t^2(t^2+1)} \\ du = \frac{dt}{t^2+1} & v = \left(-2\arctan t - \frac{2}{t}\right) dx \end{array} \right| =$$
$$= -2\arctan^2 t - \frac{2\arctan t}{t} + 2\int \frac{\arctan t}{t^2+1} + 2\int \frac{dt}{t(t^2+1)} =$$
$$= -2\arctan^2 t - \frac{2\arctan t}{t} + \arctan^2 t + 2\int \frac{dt}{t(t^2+1)} =$$
$$= -\arctan^2 t - \frac{2\arctan t}{t} + 2\int \left(\frac{1}{t} - \frac{t}{t^2+1}\right) dt =$$
$$= -\arctan^2 t - \frac{2\arctan t}{t} + 2\ln|t| - \ln|t^2+1| + C =$$
$$= -\arctan^2 e^{\frac{x}{2}} - 2e^{-\frac{x}{2}} \arctan e^{\frac{x}{2}} + x - \ln|e^x+1| + C$$

18.105

$$\int \frac{\arcsin x dx}{x^2} = \left| \begin{array}{ll} u = \arcsin x & dv = \frac{dx}{x^2} \\ du = \frac{dx}{\sqrt{1-x^2}} & v = -\frac{1}{x} \end{array} \right| = -\frac{\arcsin x}{x} + \int \frac{dx}{x\sqrt{1-x^2}} =$$
$$= -\frac{\arcsin x}{x} - \ln\left|\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right| + C$$
$$\int \frac{dx}{x\sqrt{1-x^2}} = \left| \begin{array}{l} t = \frac{1}{x} \\ \frac{1}{t} = x \\ -\frac{dt}{t^2} = dx \end{array} \right| = -\int \frac{dt}{t\sqrt{1-\frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{t^2-1}} =$$
$$= -\ln|t + \sqrt{t^2-1}| + C = -\ln\left|\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right| + C$$

18.106

$$\int \frac{\arcsin e^x}{e^x}dx = \left| \begin{array}{l} t = e^x \\ \ln t = x \\ \frac{dt}{t} = dx \end{array} \right| = \int \frac{\arcsin t}{t^2}dt = \left| \begin{array}{ll} u = \arcsin t & dv = \frac{dt}{t^2} \\ du = \frac{dx}{\sqrt{1-t^2}} & v = -\frac{1}{t} \end{array} \right| =$$
$$= -\frac{\arcsin t}{t} + \int \frac{dt}{t\sqrt{1-t^2}} = -\frac{\arcsin t}{t} - \ln\left|\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1}\right| + C =$$
$$= -e^{-x} \arcsin e^x - \ln|e^{-x} + \sqrt{e^{-2x} - 1}| + C$$

18.107

$$\int x^3\arctan x dx = \left| \begin{array}{ll} u=\arctan x & dv=x^3dx \\ du=\frac{dx}{x^2+1} & v=\frac{1}{4}x^4 \end{array}\right| = \frac{1}{4}x^4\arctan x - \frac{1}{4}\int\frac{x^4}{x^2+1}dx =$$
$$= \frac{1}{4}x^4\arctan x - \frac{1}{4}\int\frac{(x^2-1)(x^2+1)+1}{x^2+1}dx =$$
$$= \frac{1}{4}x^4\arctan x - \frac{1}{12}x^3 + \frac{1}{4}x - \frac{1}{4}\arctan x + C = \frac{1}{4}(x^4-1)\arctan x - \frac{1}{12}x(x^2-3)+C$$

18.108

$$\int(2x+3)\arccos(2x-3)dx = \left| \begin{array}{l} t=2x-3 \\ \frac{1}{2}dt=dx \end{array}\right| = \frac{1}{2}\int(t+6)\arccos t dt =$$
$$= \left| \begin{array}{ll} u=\arccos t & dv=(t+6)dt \\ du=-\frac{dt}{\sqrt{1-t^2}} & v=\frac{1}{2}t^2+6t \end{array}\right| = (\frac{1}{4}t^2+3t)\arccos t + \frac{1}{4}\int\frac{t^2+12t}{\sqrt{1-t^2}} =$$
$$= (\frac{1}{4}t^2+3t)\arccos t + \frac{1}{4}\int\frac{t^2-1+1+12t}{\sqrt{1-t^2}} =$$
$$= (\frac{1}{4}t^2+3t)\arccos t - \frac{1}{4}\int\sqrt{1-t^2}dt + \frac{1}{4}\int\frac{dt}{\sqrt{1-t^2}} + \int\frac{3t}{\sqrt{1-t^2}} =$$
$$= (\frac{1}{4}t^2+3t)\arccos t - \frac{1}{8}\arcsin t - \frac{1}{8}t\sqrt{1-t^2} + \frac{1}{4}\arcsin t - 3\sqrt{1-t^2} + C =$$
$$= (\frac{1}{4}(2x-3)^2+3(2x-3))\arccos(2x-3) - \frac{1}{8}\arcsin(2x-3) - \frac{1}{8}(2x-3)\sqrt{1-(2x-3)^2} + \frac{1}{4}\arcsin(2x-3) - 3\sqrt{}$$

18.109

$$\int\frac{x\arctan x}{\sqrt{1+x^2}}dx = \left| \begin{array}{ll} u=\arctan x & dv=\frac{x}{\sqrt{1+x^2}}dx \\ du=\frac{dx}{x^2+1} & v=\sqrt{x^2+1} \end{array}\right| = \sqrt{x^2+1}\arctan x - \int\frac{dx}{\sqrt{x^2+1}} =$$
$$= \sqrt{x^2+1}\arctan x - \ln|x+\sqrt{x^2+1}| + C$$

18.110

$$\int\sqrt{1-x^2}\arcsin x dx = \left| \begin{array}{ll} u=\arcsin x & dv=\sqrt{1-x^2}dx \\ du=\frac{dx}{\sqrt{1-x^2}} & v=\frac{1}{2}\arcsin x+\frac{1}{2}x\sqrt{1-x^2} \end{array}\right| =$$
$$= \frac{1}{2}\arcsin^2 x + \frac{1}{2}x\sqrt{1-x^2}\arcsin x - \frac{1}{2}\int\frac{\arcsin x}{\sqrt{1-x^2}} - \frac{1}{2}\int x dx =$$
$$= \frac{1}{2}\arcsin^2 x + \frac{1}{2}x\sqrt{1-x^2}\arcsin x - \frac{1}{4}\arcsin^2 x - \frac{1}{4}x^4 + C =$$
$$= \frac{1}{4}\arcsin^2 x + \frac{1}{2}x\sqrt{1-x^2}\arcsin x - \frac{1}{4}x^4 + C$$

18.111

$$\int x(1+x^2)\arctan x dx = \left| \begin{array}{ll} u=\arctan x & dv=(x+x^3)dx \\ du=\frac{dx}{x^2+1} & v=\frac{1}{2}x+\frac{1}{4}x^4 \end{array}\right| =$$
$$= (\frac{1}{2}x+\frac{1}{4}x^4)\arctan x - \frac{1}{4}\int\frac{x^4+2x}{x^2+1} = (\frac{1}{2}x+\frac{1}{4}x^4)\arctan x - \frac{1}{4}\int\frac{(x^2-1)(x^2+1)+1+2x}{x^2+1}dx =$$

$$= (\frac{1}{2}x + \frac{1}{4}x^4)\arctan x - \frac{1}{4}\int (x^2-1)dx - \frac{1}{4}\int \frac{dx}{x^2+1} - \frac{1}{4}\int \frac{2x}{x^2+1}dx =$$
$$= (\frac{1}{2}x + \frac{1}{4}x^4)\arctan x - \frac{1}{12}x^3 + \frac{1}{4}x - \frac{1}{4}\arctan x - \frac{1}{4}\ln|x^2+1| + C =$$
$$= (\frac{1}{4}x^4 + \frac{1}{2}x - \frac{1}{4})\arctan x - \frac{1}{12}x^3 + \frac{1}{4}x - \frac{1}{4}\ln|x^2+1| + C$$

18.112

$$\int \arcsin\frac{2\sqrt{x}}{1+x}dx = \left| \begin{array}{ll} u = \arcsin\frac{2\sqrt{x}}{1+x} & dv = dx \\ du = -\frac{\sqrt{x}}{x(x+1)}dx & v = x \end{array} \right| = x\arcsin\frac{2\sqrt{x}}{1+x} + \int \frac{\sqrt{x}}{x+1} =$$
$$= x\arcsin\frac{2\sqrt{x}}{1+x} + 2\sqrt{x} - 2\arctan(\sqrt{x}) + C$$
$$\int \frac{\sqrt{x}}{x+1} = \left| \begin{array}{l} t^2 = x \\ 2tdt = dx \end{array} \right| = 2\int \frac{t^2}{t^2+1}dt = 2\int \frac{t^2+1-1}{t^2+1}dt = 2t - 2\arctan t =$$
$$= 2\sqrt{x} - 2\arctan(\sqrt{x}) + C$$

18.3 § Całki funkcji wykładniczych i logarytmicznych.

18.118

$$\int (e^{3x} + \sqrt{e^x})dx = \frac{1}{3}e^{3x} + 2\sqrt{e^x} + C$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C, \text{ gdzie } a \neq 0$$

18.119

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \left| \begin{array}{l} t = e^x + e^{-x} \\ dt = (e^x - e^{-x})dx \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C = \ln(e^x + e^{-x}) + C$$

18.120

$$\int \frac{dx}{e^{2x}-1} = \left| \begin{array}{l} t = e^{2x} \\ \frac{1}{2}\ln t = x \\ \frac{dt}{2t} = dx \end{array} \right| = \frac{1}{2}\int \frac{dt}{t(t-1)} = \frac{1}{2}\left(\int \frac{dt}{t-1} - \int \frac{dt}{t}\right) =$$
$$= \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|e^{2x}+1| - x + C$$

18.121

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \left| \begin{array}{l} t = e^x \\ dt = e^x dx \end{array} \right| = \int \frac{dt}{t^2+1} = \arctan(t) + C =$$
$$= \arctan(e^x) + C$$

18.122

$$\int \sqrt{e^x+1}dx = \left| \begin{array}{l} t=\sqrt{e^x+1} \\ t^2=e^x+1 \\ 2tdt=e^xdx \end{array} \right| = \int \frac{2t^2}{t^2-1}dt = \int \frac{2(t^2-1)+2}{t^2-1}dt =$$
$$= \int 2dt + 2\int \frac{dt}{t^2-1} = 2t + \int \frac{dt}{t-1} - \int \frac{dt}{t+1} = 2t + \ln\left|\frac{t-1}{t+1}\right| + C =$$
$$= 2\sqrt{e^x+1} + \ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right| + C$$
$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

18.123

$$\int \frac{e^x-1}{e^x+1}dx = \int \frac{e^x+1-2}{e^x+1}dx = \int dx - 2\int \frac{dx}{e^x+1} = x - 2x + 2\ln|e^x+1| + C =$$
$$= 2\ln|e^x+1| - x + C$$
$$\int \frac{dx}{e^x+1} = \left| \begin{array}{l} t=e^x \\ \ln t = x \\ \frac{dt}{t}=dx \end{array} \right| = \int \frac{dt}{t(t+1)} = \int \frac{dt}{t} - \int \frac{dt}{t+1} = \ln t - \ln|t+1| =$$
$$= x - \ln|e^x+1| + C$$

18.124

$$\int \frac{dx}{\sqrt{3+2e^x}} = \left| \begin{array}{l} t=\sqrt{3+2e^x} \\ t^2=2e^x+3 \\ \frac{t^2-3}{2}=e^x \\ tdt=e^xdx \end{array} \right| = \int \frac{2dt}{t^2-3} = -2\int \frac{dt}{(\sqrt{3})^2-t^2} =$$
$$= -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C = -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{3+2e^x}}{\sqrt{3}-\sqrt{3+2e^x}}\right| + C$$

18.125

$$\int e^x\sqrt{1+e^x}dx = \left| \begin{array}{l} t=1+e^x \\ dt=e^xdx \end{array} \right| = \int \sqrt{t}dt = \frac{2}{3}t^{\frac{3}{2}} + C = \frac{2}{3}\sqrt{(e^x+1)^3} + C$$

18.126

$$\int \frac{e^x}{(e^x-1)^2}dx = \left| \begin{array}{l} t=e^x-1 \\ dt=e^xdx \end{array} \right| = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{e^x-1} + C$$

18.127

$$\int (e^x+e^{-x})^2dx = \int (e^{2x}+2+e^{-2x})dx = \frac{1}{2}(e^{2x}-e^{-2x}) + 2x + C$$

18.128

$$\int \frac{e^x}{e^x+5}dx = \int \frac{(e^x+5)'}{e^x+5}dx = \ln|e^x+5| + C$$

18.129

$$\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx = \int \frac{4e^{2x}+6}{9e^{2x}-4}dx = \left| \begin{array}{l} t = e^x \\ \ln t = x \\ \frac{dt}{t} = dx \end{array} \right| = \int \frac{4t^2+6}{t(9t^2-4)}dt = \int \frac{4t^2+6}{t(3t-2)(3t+2)}dt = ...$$

rozkład na ułamki proste:

$$\frac{4t^2+6}{t(3t-2)(3t+2)} \equiv \frac{A}{t} + \frac{B}{3t-2} + \frac{C}{3t+2}$$
$$4t^2+6 \equiv A(9t^2-4) + Bt(3t+2) + Ct(3t-2)$$
$$4t^2+6 \equiv (9A+3B+3C)t^2 + (2B-2C)t + (-4A)$$
$$\begin{cases} 9A+3B+3C = 4 \\ 2B-2C = 0 \\ -4A = 6 \end{cases}$$
$$\begin{cases} A = -\frac{3}{2} \\ B = \frac{35}{12} \\ C = \frac{35}{12} \end{cases}$$
$$... = -\frac{3}{2}\int \frac{dt}{t} + \frac{35}{12}\int \frac{dt}{3t-2} + \frac{35}{12}\int \frac{dt}{3t+2} = -\frac{3}{2}\ln|t| + \frac{35}{36}\ln|9t^2-4| + C =$$
$$= -\frac{3}{2}x + \frac{35}{36}\ln|9e^{2x}-4| + C$$

18.130

$$\int \frac{dx}{e^x+e^{2x}} = \left| \begin{array}{l} t = e^x \\ \ln t = x \\ \frac{dt}{t} = dx \end{array} \right| = \int \frac{dt}{t^2(t+1)}$$

rozkład na ułamki proste:

$$\frac{1}{t^2(t+1)} \equiv \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$
$$1 \equiv At(t+1) + B(t+1) + Ct^2$$
$$1 \equiv (A+C)t^2 + (A+B)t + B$$
$$\begin{cases} A+C = 0 \\ A+B = 0 \\ B = 1 \end{cases}$$
$$\begin{cases} A = -1 \\ B = 1 \\ C = 1 \end{cases}$$
$$... = \int \frac{-dt}{t} + \int \frac{dt}{t^2} + \int \frac{dt}{t+1} = -\ln|t| - \frac{1}{t} + \ln|t+1| + C =$$

$= -x - e^{-x} + \ln|e^x + 1| + C$

18.131

$$\int \frac{e^x}{(e^x+a)^n}dx$$

dla

$n = 1$

$$\int \frac{e^x}{e^x+a}dx = \ln|e^x+a| + C$$

dla

$n \in N_+ - \{1\}$

$$\int \frac{e^x}{(e^x+a)^n}dx = \left| \begin{array}{l} t = e^x + a \\ dt = e^x dx \end{array} \right| = \int \frac{dt}{t^n} = -\frac{1}{(n-1)t^{n-1}} + C = -\frac{1}{(n-1)(e^x+a)^{n-1}} + C$$

18.132

$$\int \frac{e^x dx}{\sqrt{3-5e^{2x}}} = \left| \begin{array}{l} t = e^x \\ dt = e^x \end{array} \right| = \int \frac{dt}{\sqrt{3-5t^2}} = \frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{1-(\sqrt{\frac{5}{3}}t)^2}} = \left| \begin{array}{l} u = \sqrt{\frac{5}{3}}t \\ \sqrt{\frac{3}{5}}du = dt \end{array} \right| =$$

$$= \frac{1}{\sqrt{5}} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{\sqrt{5}} \arcsin u + C = \frac{1}{\sqrt{5}} \arcsin(\sqrt{\frac{5}{3}}e^x) + C$$

18.133

$$\int \frac{dx}{\sqrt{e^{2x}+4e^x+1}} = \left| \begin{array}{l} t = e^x \\ \ln t = x \\ \frac{dt}{t} = dx \end{array} \right| = \int \frac{dt}{t\sqrt{t^2+4t+1}} = \left| \begin{array}{l} u = \frac{1}{t} \\ \frac{1}{u} = t \\ \frac{-du}{u^2} = dt \end{array} \right| = -\int \frac{du}{u^2\sqrt{\frac{1}{u^2}+\frac{4}{u}+1}} =$$

$$= -\int \frac{du}{\sqrt{u^2+4u+1}} = -\int \frac{du}{\sqrt{(u+2)^2-3}} = -\ln|u+2+\sqrt{u^2+4u+1}| + C =$$

$$= -\ln|e^{-x}+2+\sqrt{e^{-2x}+4e^{-x}+1}| + C$$

18.134

$$\int x^3e^{-x}dx = \left| \begin{array}{ll} u = x^3 & dv = e^{-x}dx \\ du = 3x^2dx & v = -e^{-x} \end{array} \right| = -x^3e^{-x} + 3\int x^2e^{-x}dx =$$

$$= \left| \begin{array}{ll} u = x^2 & dv = e^{-x}dx \\ du = 2xdx & v = -e^{-x} \end{array} \right| = -x^3e^{-x} - 3x^2e^{-x} + 6\int xe^{-x}dx = \left| \begin{array}{ll} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{array} \right| =$$

$$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} + 6\int e^{-x}dx = -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + C$$

18.135

$$\int \frac{dx}{x\ln x} = \left| \begin{array}{l} t = \ln x \\ dt = \frac{dx}{x} \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C = \ln|\ln x| + C$$

18.136

$$\int \ln(x^2+1)dx = \left| \begin{array}{ll} u = \ln(x^2+1) & dv = dx \\ du = \frac{2x}{x^2+1}dx & v = x \end{array} \right| = x\ln(x^2+1) + \int \frac{2x^2}{x^2+1}dx =$$
$$= x\ln(x^2+1) + \int 2dx - 2\int \frac{dx}{x^2+1} = x\ln(x^2+1) + 2x - 2\arctan(x) + C$$

18.137

$$\int (\ln|x|)^2 dx = \left| \begin{array}{ll} u = \ln^2|x| & dv = dx \\ du = \frac{2\ln|x|}{x}dx & v = x \end{array} \right| = x\ln^2|x| - 2\int \ln|x| = \left| \begin{array}{ll} u = \ln|x| & dv = dx \\ du = \frac{dx}{x} & v = x \end{array} \right| =$$
$$= x\ln^2|x| - 2x\ln|x| + 2\int dx = x\ln^2|x| - 2x\ln|x| + 2x + C$$

18.138

$$\int \ln(x+\sqrt{x^2+1})dx = \left| \begin{array}{ll} u = \ln(x+\sqrt{x^2+1}) & dv = dx \\ du = \frac{1}{\sqrt{x^2+1}} & v = x \end{array} \right| =$$
$$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}dx = x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + C$$

18.139

$$\int \ln|2+5x|dx = \left| \begin{array}{ll} u = \ln|2+5x| & dv = dx \\ du = \frac{5dx}{2+5x} & v = x \end{array} \right| = x\ln|2+5x| - \int \frac{5x}{2+5x}dx =$$
$$= x\ln|2+5x| - \int \frac{2+5x-2}{2+5x}dx = x\ln|2+5x| - \int dx + 2\int \frac{dx}{2+5x} =$$
$$= x\ln|2+5x| - x + \frac{2}{5}\ln|2+5x| + C = (x+\frac{2}{5})\ln|2+5x| - x + C$$

18.140

$$\int \frac{dx}{x(1+\ln^2|x|)} = \left| \begin{array}{l} t = \ln|x| \\ dt = \frac{dx}{x} \end{array} \right| = \int \frac{dt}{1+t^2} = \arctan(t) + C = \arctan(\ln|x|) + C$$

18.141

$$\int x^{-2}\ln|x|dx = \left| \begin{array}{ll} u = \ln|x| & dv = \frac{dx}{x^2} \\ du = \frac{dx}{x} & v = -\frac{1}{x} \end{array} \right| = -\frac{\ln|x|}{x} + \int \frac{dx}{x^2} = -\frac{\ln|x|}{x} - \frac{1}{x} + C$$

18.142

$$\int (4+3x)^2\ln|x|dx = \int (9x^2+24x+16)\ln|x|dx = \left| \begin{array}{ll} u = \ln|x| & dv = (9x^2+24x+16)dx \\ du = \frac{dx}{x} & v = 3x^3+12x^2+16x \end{array} \right| =$$
$$= (3x^3+12x^2+16x)\ln|x| - \int (3x^2+12x+16)dx =$$
$$= (3x^3+12x^2+16x)\ln|x| - x^3 - 6x^2 - 16x + C$$

18.143

$$\int x^3\ln(x^2+3)dx=\frac{1}{2}\int 2x(x^2+3-3)\ln(x^2+3)=\begin{vmatrix} t=x^2+3 \\ dt=2xdx \end{vmatrix}=$$
$$=\frac{1}{2}\int(t-3)\ln t dt=\begin{vmatrix} u=\ln t & dv=(t-3)dt \\ du=\frac{dt}{t} & v=\frac{1}{2}t^2-3t \end{vmatrix}=$$
$$=\frac{1}{4}(t^2-6t)\ln t-\frac{1}{2}\int(\frac{1}{2}t-3)dt=\frac{1}{4}(t^2-6t)\ln t-\frac{1}{8}t^2+\frac{3}{2}t+C=$$
$$=\frac{1}{4}(x^4-9)\ln|x^2+3|-\frac{1}{8}(x^2+3)^2+\frac{3}{2}(x^2+3)+C$$

18.144

$$\int xa^x dx,\ a>1$$
$$=\begin{vmatrix} u=x & dv=a^x dx \\ du=dx & v=\frac{a^x}{\ln a} \end{vmatrix}=\frac{xa^x}{\ln a}-\frac{1}{\ln a}\int a^x dx=\frac{xa^x}{\ln a}-\frac{a^x}{\ln^2 a}+C$$